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OPTIMUM CRUISE PERFORMANCE

HIGH SPEED AEROPERFORMANCE BRANCH
AEROMECHANICS DIVISION

NOVEMBER 1978

TECHNICAL REPORT AFFDL-TR-78-131
Final Report for Period June to August 1977

Approved for public release; distribution unlimited.

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
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This technical report has been reviewed and is approved for publication.

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This report considers the cruise performance of a jet-propelled aircraft at high speed. The two problems of cruise with maximum range and with maximum endurance are analyzed. In each problem, for any given aircraft aerodynamics and engine characteristics, the equation for determining the optimum Mach number for cruise is derived. For maximum range, there exists an optimum cruise altitude. For the maximum endurance prob- OVER

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lem, the effect of the altitude on performance is negligible. It is shown that in both problems, constant Mach number cruise is a satisfactory flying technique. In the true optimum solution the optimum Mach number slowly decreases along the flight path. In this case, the singular thrust control is obtained explicitly as function of the Mach number.

FOREWORD

This report was prepared by Professor N.X. Vinh of the University of Michigan, Ann Arbor, Michigan. This research was accomplished under the American Society for Engineering Education program. Dr. L. Earl Miller of the Air Force Flight Dynamics Laboratory, Wright-Patterson, Air Force Base, Ohio, served as Professor Vinh's Air Force colleague. The work was accomplished under work unit 24040709 Flight Performance Analysis and Design Methods. The period of the effort was from June to August 1977.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

a	Speed of sound
A	Modified adjoint variable, Eq. (147)
A_E	Function for the endurance problem, Eq. (56)
A_R	Function for the range problem, Eq. (55)
b	Function of M.Eq. (207) for the range problem and Eq. (227) for the endurance problem
B	Modified adjoint variable, Eq. (147)
c	Specific fuel consumption
C	Modified adjoint variable, Eq. (147)
$C_1,\ldots,C_5$	Constants of integration
C_D	Drag coefficient
C_{D_o}	Zero-lift drag coefficient
C_{D_1}	Drag coefficient in incompressible flow
C_L	Lift coefficient
D	Drag force
D	Modified adjoint variable, Eq. (147)
E_{max}	Maximum lift-to-drag ratio
g	Acceleration of the gravity
k	Ratio of specific heats of ambient air
k_c	Specific fuel consumption coefficient, Eq. (41)
K	Induced-drag coefficient
K_1	Value of K in incompressible flow
K_1, K_2	Functions of Mach number. Eqs. (182) for the range problem and Eqs. (213) for the endurance problem.

LIST OF SYMBOLS (CONTINUED)

K_B	Constant coefficient, Eq. (23)
K_c	Modified specific fuel consumption coefficient, Eq. (17)
L	Lift force
m	Mass of the aircraft
M	Mach number
M_1	Mach number beyond which compressibility takes effect
n	Coefficient in atmospheric density-pressure relation
p	Atmospheric pressure
p	With subscripts x, η, M, γ, w, θ. Adjoint variables
P	Function of M and ω. Eq. (200) for the range problem and Eq. (222) for the endurance problem
Q	Function of M and ω. Eq. (201) for the range problem and Eq. (223) for the endurance problem
S	Aerodynamic reference area
t	Time
T	Thrust magnitude
V	Speed
w	Dimensionless weight Eq. (17)
W	Weight of the aircraft
x	Dimensionless longitudinal distance, Eq. (17)
X	Longitudinal distance
z	Dimensionless altitude, Eq. (17)
Z	Altitude

LIST OF SYMBOLS (CONTINUED)

Greek Symbols

α	Angle of attack
β	Inverse of atmosphere height scale
γ	Flight path angle
δ	Pressure ratio, Eq. (40)
ε	Angle between thrust vector and reference line
λ	Coefficient in atmospheric density-pressure relation
η	Dimensionless inverse pressure, Eq. (17)
ω	Dimensionless wing loading, Eq. (18)
Ω	Modified dimensionless wing loading, Eq. (95)
ρ	Atmospheric density
θ	Dimensionless time, Eq. (17)
τ	Dimensionless thrust magnitude, Eq. (17)

Superscript

*	Condition of maximum lift-to-drag ratio

Subscripts

i	Initial condition
f	Final condition
*	Condition at the tropopause
$(\ )_M$	Denotes logarithmic derivative, Eq. (36)

SECTION I

INTRODUCTION

In this report we shall consider the cruise performance of a jet-propelled aircraft at high speed. A characteristic of the high-speed performance problem is that the aerodynamic and engine controls are strongly influenced by the altitude and the Mach number. Hence, in performance analysis, it is nearly impossible to obtain general results which are unrestrictedly valid for all flight missions. For that reason we shall restrict our investigation to the cruise performance problem.

In general, cruise performance is characterized by flight in the vertical plane at small flight path angle and nearly constant engine setting. This, in turn, leads to slow variations in the speed and altitude.

Typical optimum cruise performance problems involve the derivation of aerodynamic and engine controls for maximum range or maximum endurance in either constant altitude flight or in missions which allow variations in the altitude. The constant altitude flight is representative of commercial flights regulated by the Federal Aviation Agency. Since maximizing the range for a given amount of fuel is obviously equivalent to minimizing the fuel to cover a prescribed distance, the effort of the author of this technical report is in line with the national policy of saving energy. On the other hand, the maximum endurance cruising flight is representative of

military peacetime missions. Maximum endurance for a given amount of fuel is also equivalent to minimizing the fuel for the prescribed peacekeeping duration of a mission.

The organization of this report is as follows. After this introductory section, the equations of motion and the problem definition are discussed in Section II. Pertinent assumptions regarding the analysis will be presented in Section III. The equations of motion using dimensionless variables will be proposed in Section IV. In Section V, the complete analytical solutions for the two problems of maximum range and maximum endurance in the case of constant altitude flight will be presented. The problem of optimum aerodynamic and engine controls for cruise performance with variations in the altitude is the subject of the next two sections. In Section VI, an engineering approach is used to derive simple optimum laws for aerodynamic and engine controls in both the maximum range and maximum endurance problems. In the next section, Section VII, the same problems are treated by optimal control theory using the full set of equations.

SECTION II

PROBLEM DEFINITION

The flight of a lifting, thrusting vehicle in a vertical plane is governed by the equations (Fig. 1)

$$
\begin{aligned}
\frac{dX}{dt} &= V \cos \gamma \\
\frac{dZ}{dt} &= V \sin \gamma \\
m \frac{dV}{dt} &= T \cos (\varepsilon + \alpha) - D - mg \sin \gamma \\
mV \frac{d\gamma}{dt} &= T \sin (\varepsilon + \alpha) + L - mg \cos \gamma \\
\frac{dm}{dt} &= - \frac{c}{g} T
\end{aligned}
\tag{1}
$$

The first two equations are the kinematic equations. The next two equations are the force equations written along the directions tangent and normal to the flight path. The last equation represents the mass flow as related to the thrust. Standard notation as defined in the nomenclature section has been used. The flight path angle γ has been selected such that γ is positive when the velocity is directed above the local horizontal plane.

The flight is controlled by varying the lift and drag forces, through the angle of attack α, and the thrust magnitude T. It is proposed to find these aerodynamic and engine controls to bring the vehicle from a certain prescribed initial state (X_i, Z_i, V_i, γ_i, m_i) to some partially prescribed final state in the first four var-

iables and a prescribed final mass m_f in such a way that a certain function of the final variables is a maximum. In the problem of maximum range, we maximize the final distance X_f while in the problem of maximum endurance we maximize the final t_f.

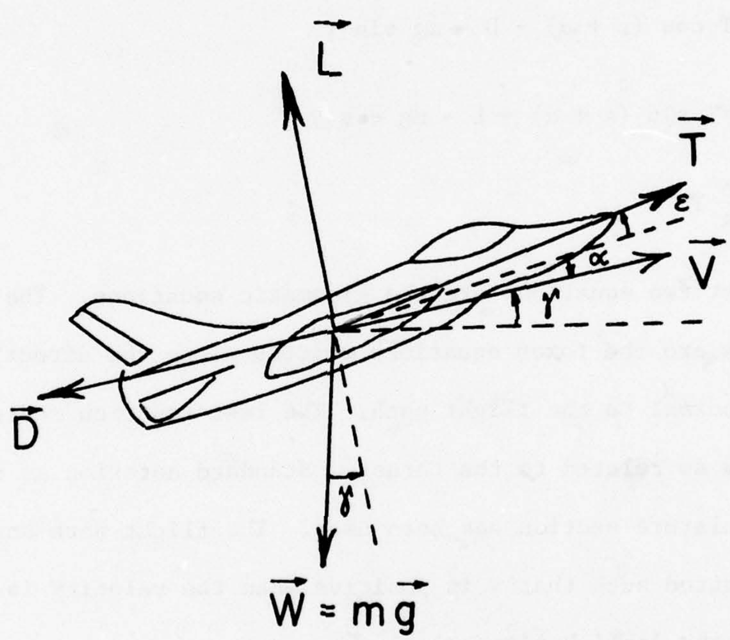

Fig. 1. Nomenclature.

SECTION III

ASSUMPTIONS

To analyze the system of equations (1), two types of assumptions are introduced. The first type consists of the modeling of the medium in which the vehicle is flying, namely in this case the earth's atmosphere, and the vehicle aerodynamics and engine characteristics. This is necessary for any mathematical treatment of a physical problem. The second type of assumptions is designed to simplify the equations of motion such that analytical manipulation is feasible. In this respect we shall neglect the terms which are small, and treat as constants the terms which are insensitive to change along the flight path. For the results to have any validity, unnecessary simplification should be avoided and accuracy of the solutions shouldn't be compromised.

The aerodynamic characteristics of the vehicle strongly depends on the Mach number defined as

$$M = \frac{V}{a} \tag{2}$$

where a is the speed of sound in the ambient atmosphere. The speed of sound is related to the atmospheric density ρ and the pressure p through the relation

$$a^2 = \frac{kp}{\rho} \tag{3}$$

where k is the ratio of the specific heats of the ambient air, con-

sidered as a constant. An approximate relation between the pressure and the density is

$$\frac{p}{p_o} = \lambda\left(\frac{\rho}{\rho_o}\right)^n \tag{4}$$

where subscript zero denotes the condition at sea level and λ and n are two constants appropriately selected. Two sets of values for λ and n will be used depending on the atmospheric layer in which the vehicle is flying. The troposphere is the first layer of the atmosphere extending from sea level to approximately 36,089 feet. Across this layer the atmospheric temperature, and hence the speed of sound, varies with the altitude. Above the limit of the troposphere, called the tropopause, and extending to approximately 66,000 feet is the stratosphere. In the stratosphere the atmospheric temperature and the speed of sound remain nearly constant. The flight will be assumed to take place in either the troposphere or the stratosphere. For the sake of continuity of the equations, it is more convenient to use the reference condition at the tropopause. Hence, we shall use the density-pressure relationship in the form

$$\frac{p}{p_*} = \left(\frac{\rho}{\rho_*}\right)^n \tag{5}$$

where subscript * denotes the condition at the tropopause. The value of n is $n = 1.235$ in the troposphere and $n = 1$ in the stratosphere.

The aerodynamic drag and lift are assumed to be of the form

$$D = \frac{1}{2} \rho S V^2 C_D$$
$$L = \frac{1}{2} \rho S V^2 C_L \tag{6}$$

where C_D and C_L are respectively the drag and the lift coefficient. It is assumed that C_D and C_L are related by the parabolic drag polar.

$$C_D = C_{D_o} + K C_L^2 \tag{7}$$

where the zero-lift drag coefficient C_{D_o} and the induced-drag coefficient K are functions of the Mach number

$$C_{D_o} = C_{D_o} (M)$$
$$K = K (M) \tag{8}$$

From experimental data, it is possible to model these functions as polynomials in M. In general, it is assumed that they are continuous functions in M with continuous first derivatives. Quite frequently, for a given Mach number, the drag polar, that is the plot of C_D versus C_L does not yield the minimum of C_D for $C_L = 0$ (Fig. 2). As shown in Ref. 1, it is remedied by a simple translation along the C_L axis. This will introduce additional terms in the pertinent results. Fortunately, in optimum cruise performance, the operating point for the angle of attack is near the point of maximum lift-to-drag ratio and not where the drag coefficient is minimum (Fig. 2). In this respect, the accuracy is not compromised by using relation (7) and evaluating C_{D_o} and K such that we have perfect fitting of

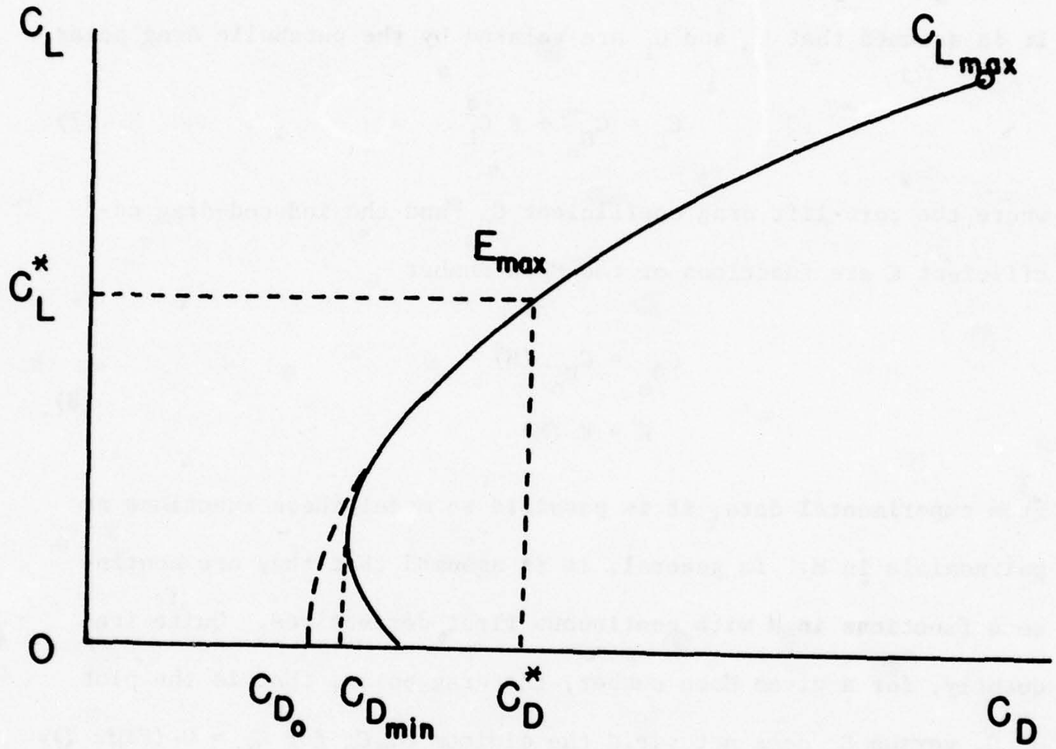

Fig. 2. Parabolic Drag Polar.

the drag polar in its upper part, especially near the point of maximum lift-to-drag ratio. This ratio defined as

$$E_{max} = \left(\frac{C_L}{C_D}\right)_{max} \tag{9}$$

is a function of the Mach number and is an important parameter in performance analysis. If relation (7) is used, then

$$E_{max} = \frac{1}{2\sqrt{KC_{D_o}}} \tag{10}$$

It is assumed that parabolic drag polar is valid up to a certain angle-of-attack called the stalling angle-of-attack. At this point, C_L reaches a maximum value

$$C_{L_{max}} = C_{L_{max}}(M) \tag{11}$$

and beyond it the lift coefficient drops drastically because of the turbulent effect due to flow separation. For all practical purposes, we can consider $C_{L_{max}}$, which is a function of the Mach number, as the limiting value for the lift coefficient.

The parameter c is called the specific fuel consumption. It is a function of the Mach number, of the engine throttling, and in the neighborhood of the tropopause it slightly depends on the altitude. Generally speaking, for subsonic cruise, there is a tradeoff between the aerodynamic characteristics and the specific fuel consumption to find the best cruise point. Since this influence has a secondary effect as compared to the influence of the Mach

number and also since the optimum value of c is usually near its minimum value, we shall assume that c is minimum and this minimum value is purely a function of the Mach number.

It is accurate to assume that the acceleration of the gravity is constant. We now turn to the second type of assumptions. In the equations (1), ε is the angle between the thrust vector and the reference line from which we measure the angle of attack α. Both ε and α are small and the analysis will be much simplified by taking in the equations

$$\begin{aligned} T \cos(\varepsilon + \alpha) &\approx T \\ T \sin(\varepsilon + \alpha) &\approx 0 \end{aligned} \tag{12}$$

This is the same as saying that the thrust vector is always aligned with the velocity vector. The assumption may not be valid in the case of flight at low speed of a delta-wing aircraft. Other simplifying assumptions will be introduced later when it becomes necessary to use them in order to obtain an explicit solution to the problem considered.

SECTION IV

DIMENSIONLESS EQUATIONS

Using the assumptions (12), we rewrite the equations of motion using a parabolic drag polar

$$\frac{dX}{dt} = V \cos \gamma$$

$$\frac{dZ}{dt} = V \sin \gamma$$

$$m \frac{dV}{dt} = T - \frac{1}{2} \rho S V^2 (C_{D_o} + K C_L^2) - mg \sin \gamma \tag{13}$$

$$mV \frac{d\gamma}{dt} = \frac{1}{2} \rho S V^2 C_L - mg \cos \gamma$$

$$\frac{dm}{dt} = - \frac{c}{g} T$$

The flight is controlled by the angle-of-attack or, equivalently the lift coefficient C_L, and the thrust magnitude T subject to the constraints

$$0 \leq C_L \leq C_{L_{max}}$$

$$0 \leq T \leq T_{max} \tag{14}$$

There exists a constraint on the load factor $n = L/W$ and it is expressed by

$$n_{min} \leq n \leq n_{max}$$

It is convenient for the analysis to write the equations of motion in dimensionless form.

The advantage of using dimensionless variables is that if the final equations can be written in a form such that they are free of the physical characteristics of the vehicle, then the results of the analysis apply to any aircraft regardless of its size, mass, aerodynamics and engine characteristics. This has been illustrated by the excellent analysis of reentry trajectories by Chapman (Ref. 2). Chapman's equations are valid for the case of gliding flight, T=0, and when the aerodynamics characteristics are independent of the Mach number. Unfortunately, this is not true for our case and hence although we can put the equations in dimensionless form they still contain the vehicle's aerodynamics and engine characteristics. Nevertheless, the results obtained are general enough such that they apply to a large class of aircraft.

Chapman's search for dimensionless variables has been by trial and error. Here we shall start with the original equations, the Eqs. (13), and use a series of rational and systematic transformations leading to a logical choice of the dimensionless variables.

First, since the aerodynamics and engine characteristics are functions of the Mach number, M is the choice of the dimensionless variable to replace the speed V. In this respect, we consider

$$\frac{dV}{dt} = a\,\frac{dM}{dt} + M\,\frac{da}{dt} \tag{15}$$

In the troposphere, a is a function of the altitude, while it is a constant in the stratosphere. Since the analysis considers

cruise performance in which the altitude either remains constant or varies slowly, we shall take $da/dt = 0$.

Next, it is convenient to scale the weight and the thrust with respect to the initial weight of the vehicle. Hence, we can define the dimensionless weight w and the dimensionless thrust τ as

$$w = \frac{W}{W_i} \ , \ \tau = \frac{T}{W_i}$$

Now, since the speed of sound a will appear in the equations through relation (15), with relations (3) and (5) we can express a in terms of either the pressure p or the density ρ of the atmosphere. Then, either p or ρ can be used to represent the altitude. If we choose p, then

$$\left(\frac{a}{a_*}\right)^2 = \left(\frac{p}{p_*}\right)\left(\frac{\rho_*}{\rho}\right) = \left(\frac{p}{p_*}\right)^{(n-1)/n}$$

where subscript * denotes the condition at the tropopause.

Since

$$\rho V^2 = \rho M^2 a^2 = kpM^2$$

we can write the equation for γ

$$Mw\left(\frac{a}{a_*}\right)\frac{a_*}{g}\ \frac{d\gamma}{dt} = \frac{kpS}{2W_i} M^2 C_L - w \cos \gamma$$

Then to eliminate kS from the equation, it is logical to define the altitude variable as

$$\eta = \frac{2W_i}{kpS}$$

In this way, since p decreases as the altitude increases, η varies in the same direction as the altitude. The equation for γ is dimensionless if the dimensionless time θ is defined as

$$\theta = \frac{gt}{a_*}$$

With the dimensionless time so defined, the equation for the range is

$$\left(\frac{a_*}{a}\right)\left(\frac{g}{a_*^2}\right)\frac{dX}{d\theta} = M \cos \gamma$$

Consequently, we should replace the longitudinal distance X and the altitude Z by their dimensionless equivalents

$$x = \frac{gX}{a_*^2} \quad , \quad z = \frac{gZ}{a_*^2}$$

Finally, noticing that

$$\frac{a}{a_*} = \left(\frac{p}{p_*}\right)^{(n-1)/2n} = \left(\frac{\eta}{\eta_*}\right)^{(1-n)/2n}$$

we can write the system (13) as

$$\frac{dx}{d\theta} = \left(\frac{\eta}{\eta_*}\right)^{(1-n)/2n} M \cos \gamma$$

$$\frac{dz}{d\theta} = \left(\frac{\eta}{\eta_*}\right)^{(1-n)/2n} M \sin \gamma$$

$$\frac{dM}{d\theta} = \frac{1}{w}\left(\frac{\eta}{\eta_*}\right)^{(n-1)/2n} [\tau - \eta^{-1}M^2(C_{D_o} + KC_L^2) - w \sin \gamma]$$

$$\frac{d\gamma}{d\theta} = \frac{1}{wM}\left(\frac{\eta}{\eta_*}\right)^{(n-1)/2n} (\eta^{-1}M^2C_L - w \cos \gamma)$$

$$\frac{dw}{d\theta} = - \frac{a_* c}{g} \tau$$

The dimensionless specific fuel consumption can then be defined as

$$k_c = \frac{a_* c}{g}$$

For flight in the stratosphere $(\eta_*)^{(1-n)/2n} = 1$ and the equations are free of the parameter η_*. To remove this term when $n \neq 1$, we define

$$B = (\eta_*)^{(1-n)/2n} = \left(\frac{2W_i}{kp_* S}\right)^{(1-n)/2n} \tag{16}$$

and include it as a coefficient of the dimensionless time θ as can be done in the equations for M and γ. For the other equations, it requires a new definition for x, z and k_c. This finally leads to the following definition

$$x = \frac{B^2 g}{a_*^2} X \quad , \; z = \frac{B^2 g}{a_*^2} Z$$

$$w = \frac{W}{W_i} \quad , \; \tau = \frac{T}{W_i} \tag{17}$$

$$\eta = \frac{2W_i}{kpS} \; , \; K_c = \frac{a_* c}{Bg} \; , \; \theta = \frac{Bg}{a_*} t$$

In some analysis, it is convenient to combine w and η by using Miele's dimensionless wing loading defined as (Ref. 3).

$$\omega = \frac{2W}{kpS} = \eta w \tag{18}$$

With the dimensionless variables as defined in Eqs. (17) we

have the dimensionless equations in their final form

$$\frac{dx}{d\theta} = \eta^{(1-n)/2n} M \cos \gamma$$

$$\frac{dz}{d\theta} = \eta^{(1-n)/2n} M \sin \gamma$$

$$\frac{dM}{d\theta} = \frac{\eta^{(n-1)/2n}}{w} [\tau - \eta^{-1} M^2 (C_{D_o} + KC_L^2) - w \sin \gamma] \tag{19}$$

$$\frac{d\gamma}{d\theta} = \frac{\eta^{(n-1)/2n}}{wM} (\eta^{-1} M^2 C_L - w \cos \gamma)$$

$$\frac{dw}{d\theta} = - K_c \tau$$

First, we notice that the two dimensionless variables z and η are not independent. Explicitly, if Z is measured from the tropopause we use an exponential of the form

$$\rho = \rho_* e^{-\beta Z} \tag{20}$$

where the constant β has different values in different atmospheric layers. Then we have

$$\left(\frac{\eta}{\eta_*}\right)^{1/n} = e^{\beta Z} \tag{21}$$

Hence, if we choose η as the altitude variable, then the equation for z can be replaced by the equation for η

$$\frac{d\eta}{d\theta} = K_B \, \eta^{(n+1)/2n} M \sin \gamma \tag{22}$$

where the constant K_B is defined as

$$K_B = \frac{n\beta a_*^2}{B^2 g} \tag{23}$$

Next, since K_c is proportional to c, this specific fuel consumption coefficient is function of the Mach number

$$K_c = K_c (M) \tag{24}$$

It is assumed that this function can be approximated by a certain polynomial. Finally, the thrust coefficient varies between its limits

$$0 \leq \tau \leq \tau_{max} \tag{25}$$

While the case $\tau = 0$ is trivial, the upper limit τ_{max} is function of both the altitude and the Mach number.

SECTION V

FLIGHT AT CONSTANT ALTITUDE

If the aircraft is required to maintain a constant altitude, $\gamma = 0$, then $d\gamma/d\theta = 0$ and we have the equilibrium condition

$$C_L = \frac{\eta w}{M^2} \tag{26}$$

The system of dimensionless equations (19) is reduced to

$$\begin{aligned}
\frac{dx}{d\theta} &= \eta^{(1-n)/2n} M \\
\frac{dM}{d\theta} &= \frac{\eta^{(n-1)/2n}}{w} \tau - \frac{\eta^{-(n+1)/2n}}{w} M^2 \left(C_{D_o} + \frac{K\eta^2 w^2}{M^4}\right) \\
\frac{dw}{d\theta} &= -K_c \tau \\
\frac{d\theta}{d\theta} &= 1
\end{aligned} \tag{27}$$

where now η is a constant parameter. The single control variable is the thrust coefficient τ subject to the constraint (25). The last of the Eq. (27), which is trivial, is introduced to treat both the maximum range and the maximum endurance problems in a single formulation. Here, it should be emphasized that K_c is only a function of the Mach number, but not of the thrust, an assumption which is generally valid for subsonic cruise.

Variational Formulation

The problem is formulated as an optimal control problem. The equations (27) are the state equations. The initial conditions are

$$\theta = 0,\ x = 0,\ M = M_i,\ w = w_i = 1 \tag{28-a}$$

It is required to find the thrust magnitude control τ, and subsequently the angle-of-attack to maintain level flight, such that at the final time, for a prescribed w_f and M_f

$$\theta = \theta_f,\ x = x_f,\ M = M_f,\ w = w_f \tag{28-b}$$

For the maximum range problem, we maximize x_f while allowing the final time θ_f to be free, and for the maximum endurance problem, the final time θ_f is to be maximized while the range x_f is free.

Using the maximum principle, we introduce the adjoint vector $\vec{p} = (p_x, p_M, p_w, p_\theta)$ to form the Hamiltonian

$$H = p_\theta + \eta^{(1-n)/2n} M p_x - \frac{\eta^{-(n+1)/2n}}{w} M^2 p_M \left(C_{D_o} + \frac{K\eta^2 w^2}{M^2}\right) + \frac{\tau}{w}\left(\eta^{(n-1)/2n} p_M - wK_c p_w\right) \tag{29}$$

where the components of $\vec{p}$ are governed by the equations

$$\frac{dp_x}{d\theta} = -\frac{\partial H}{\partial x} = 0$$

$$\frac{dp_M}{d\theta} = -\frac{\partial H}{\partial M} = -\eta^{(1-n)/2n} p_x + \frac{\eta^{-(n+1)/2n}}{w} p_M \left(2MC_{D_o} + M^2 C_{D_o}' + \frac{K'\eta^2 w^2}{M^2} - \frac{2K\eta^2 w^2}{M^3}\right) + K_c' p_w \tau \tag{30}$$

$$\frac{dp_w}{d\theta} = -\frac{\partial H}{\partial w} = \eta^{-(n+1)/2n} M^2 p_M \left(-\frac{C_{D_o}}{w^2} + \frac{K\eta^2}{M^4}\right) + \frac{\eta^{(n-1)/2n}}{w^2} p_M \tau$$

$$\frac{dp_\theta}{d\theta} = - \frac{\partial H}{\partial \theta} = 0$$

where the prime denotes derivatives taken with respect to M.

The problem is solved by integrating the state and adjoint equations, subject to the end conditions (28-a) and (28-b) while selecting, at each instant θ, the thrust control τ such that the Hamiltonian H is an absolute maximum.

Optimum Aerodynamics and Engine Thrust Controls

The problem has a number of integrals

$$p_\theta = C_1$$

$$p_x = C_2 \tag{31}$$

$$H = 0$$

where C_1 and C_2 are constants of integration.

In the maximum range problem, if the final time is free, $C_1 = 0$.

In the maximum endurance problem, if the final distance is free, $C_2 = 0$.

To select the optimum thrust control, we consider the function

$$\Theta = \eta^{(n-1)/2n} p_M - K_c w p_w \tag{32}$$

called the switching function. Then, to maximize H,

If $\Theta > 0$ we choose $\tau = \tau_{max}$

If $\Theta < 0$ we choose $\tau = 0$

If $\Theta = 0$ for a finite time interval $\theta\varepsilon[\theta_1,\theta_2]$ we choose τ = variable. The optimum trajectory is a combination of subarcs:

Maximum thrust arc in which $\tau = \tau_{max}$

Coasting arc in which $\tau = 0$

Sustaining arc in which τ = variable

It is assumed that maximum thrust arc and coasting arc for both the maximum range and maximum endurance problems are of short duration. This rather intuitive assessment will be confirmed later by rigorous analysis.

Along the main portion of sustaining flight, we have for all time θ

$$\eta^{(n-1)/2n} p_M = K_c w p_w \tag{33}$$

By taking the derivative of this equation with respect to θ,

$$\eta^{(n-1)/2n} \frac{dp_M}{d\theta} = K_c p_w \frac{dw}{d\theta} + K_c' w p_w \frac{dM}{d\theta} + K_c w \frac{dp_w}{d\theta} \tag{34}$$

Using the state and adjoint equations, together with relation (33) in Eq. (34), we have the condition for optimum sustaining flight

$$\begin{aligned} C_2 \eta^{1/n} M = \frac{M^2 p_M}{w} \{ & C_{D_o} [2 + \eta^{(1-n)/2n} MK_c + (C_{D_o} K_c)_M] \\ & - \frac{K\eta^2 w^2}{M^4} [2 + \eta^{(1-n)/2n} MK_c - (KK_c)_M] \} \end{aligned} \tag{35}$$

where, following Miele (Ref. 3), to simplify the notation, we have defined the logarithmic derivatives

$$C_{D_{o_M}} = \frac{d(\mathrm{Log}\ C_{D_o})}{d(\mathrm{Log}\ M)} = \frac{M}{C_{D_o}} \frac{dC_{D_o}}{dM}$$

$$K_M = \frac{d(\mathrm{Log}\ K)}{d(\mathrm{Log}\ M)} = \frac{M}{K}\frac{dK}{dM} \tag{36}$$

$$K_{c_M} = \frac{d(\mathrm{Log}\ K_c)}{d(\mathrm{Log}\ M)} = \frac{M}{K_c}\frac{dK_c}{dM}$$

On the other hand, we can write the Hamiltonian integral H = 0, for the case of sustaining flight as

$$C_1\eta^{(n+1)/2n} + C_2\eta^{1/n}M = \frac{M^2 p_M}{w}\left(C_{D_o} + \frac{K\eta^2 w^2}{M^4}\right) \tag{37}$$

By eliminating the adjoint p_M between the Eqs. (35) and (37) we have the relation between the weight w and the Mach number M for the general case of sustaining flight at constant altitude.

For the case of maximum range, free final time, we have C_1 = 0. The elimination of p_M leads to relation

$$\omega = M^2\sqrt{\frac{C_{D_o}}{K}}\sqrt{\frac{1 + \eta^{(1-n)/2n}MK_c + (C_{D_o}K_c)_M}{3 + \eta^{(1-n)/2n}MK_c - (KK_c)_M}} \tag{38}$$

For the case of maximum endurance, free range, we have C_2 = 0. This gives

$$\omega = M^2\sqrt{\frac{C_{D_o}}{K}}\sqrt{\frac{2 + \eta^{(1-n)/2n}MK_c + (C_{D_o}K_c)_M}{2 + \eta^{(1-n)/2n}MK_c - (KK_c)_M}} \tag{39}$$

For a prescribed flight altitude, Eq. (38), upon solving, gives the optimum Mach number as function of the weight in the case of maximum range. Likewise, Eq. (39) gives the Mach number as function of the weight in the case of maximum endurance.

To assess the influence of the altitude and weight on the Mach number, we define the pressure ratio

$$\delta = \frac{p}{p_*} = \frac{\eta_*}{\eta} \tag{40}$$

and the specific fuel consumption coefficient

$$k_c = \frac{a_* c}{g} = \eta_*^{(1-n)/2n} K_c \tag{41}$$

The Eqs. (38) and (39) have their final form:

For maximum range

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} \sqrt{\frac{1 + \delta^{(n-1)/2n} M k_c + (C_{D_o} k_c)_M}{3 + \delta^{(n-1)/2n} M k_c - (K k_c)_M}} \tag{42}$$

For maximum endurance

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} \sqrt{\frac{2 + \delta^{(n-1)/2n} M k_c + (C_{D_o} k_c)_M}{2 + \delta^{(n-1)/2n} M k_c - (K k_c)_M}} \tag{43}$$

If point performance analysis is considered, by maximizing at each instant the rate of change of the distance for the maximum range problem, and minimizing the mass flow for the endurance problem, we shall come up with two equations identical to the Eqs. (42)

and (43) without the term $\delta^{(n-1)/2n} Mk_c$. This term is the correctional term due to the slow decrease of the Mach number. In general, it is a small term.

The lift coefficient is obtained from the equilibrium condition, Eq. (26), written in terms of the Mach number as

$$\omega = M^2 C_L \tag{44}$$

Hence, from Eqs. (42) and (43), we have:

For maximum range

$$C_L = C_L^* \sqrt{\frac{1 + \delta^{(n-1)/2n} Mk_c + (C_{D_o} K_c)_M}{3 + \delta^{(n-1)/2n} Mk_c - (Kk_c)_M}} \tag{45}$$

For maximum endurance

$$C_L = C_L^* \sqrt{\frac{2 + \delta^{(n-1)/2n} Mk_c + (C_{D_o} k_c)_M}{2 + \delta^{(n-1)/2n} Mk_c + (Kk_c)_M}} \tag{46}$$

where

$$C_L^* = \sqrt{\frac{C_{D_o}}{K}} \tag{47}$$

is the lift coefficient corresponding to maximum lift-to-drag ratio.

In terms of the lift-to-drag ratio, we have:

For maximum range

$$\frac{C_L}{C_D} = E_{max} \frac{[1+\delta^{(n-1)2n} Mk_c+(C_{D_o} k_c)_M]^{1/2}[3+\delta^{(n-1)/2n} Mk_c-(Kk_c)_M]^{1/2}}{2+\delta^{(n-1)/2n} Mk_c+(1/2)(C_{D_o}/K)_M} \tag{48}$$

For maximum endurance

$$\frac{C_L}{C_D} = E_{max} \frac{[2+\delta^{(n-1)/2n_{Mk_c}}+(C_{D_o}k_c)_M]^{1/2}[2+\delta^{(n-1)/2n_{Mk_c}}-(Kk_c)_M]^{1/2}}{2+\delta^{(n-1)/2n_{Mk_c}}+(1/2)(C_{D_o}/K)_M} \tag{49}$$

From an operational point of view, if the instantaneous Mach number is used as control variable, then it is obtained from either Eq. (42) or Eq. (43), depending on the case. The thrust is then adjusted to maintain constant altitude flight. Analytically, since the Mach number varies slowly along the flight, approximate value for the thrust is T=D.

This gives

$$\eta\tau = M^2(C_{D_o} + \frac{k\omega^2}{M^4}) \tag{50}$$

Hence, for the case of maximum range

$$\frac{2T}{kpS} = \left[\frac{2+\delta^{(n-1)/2n_{Mk_c}}+(1/2)(C_{D_o}/K)_M}{3+\delta^{(n-1)/2n_{Mk_c}}-(Kk_c)_M}\right] C_D^* M^2 \tag{51}$$

For the case of maximum endurance, we have the expression for the thrust

$$\frac{2T}{kpS} = \left[\frac{2+\delta^{(n-1)/2n_{Mk_c}}+(1/2)(C_{D_o}/K)_M}{2+\delta^{(n-1)/2n_{Mk_c}}-(Kk_c)_M}\right] C_D^* M^2 \tag{52}$$

where

$$C_D^* = 2\,C_{D_o} \tag{53}$$

is the drag coefficient corresponding to maximum lift-to-drag ratio.

If exact variational analysis is considered, then along the optimum trajectory for sustaining flight, the Eq. (42), or (43) must be constantly satisfied, depending on the case.

Hence, we write

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} A^{1/2} \tag{54}$$

with

$$A = A_R = \frac{1+\delta^{(n-1)/2n} Mk_c + (C_{D_o} k_c)_M}{3+\delta^{(n-1)/2n} Mk_c - (Kk_c)_M} \tag{55}$$

for the case of maximum range and

$$A = A_E = \frac{2+\delta^{(n-1)/2n} Mk_c + (C_{D_o} k_c)_M}{2+\delta^{(n-1)/2n} Mk_c - (Kk_c)_M} \tag{56}$$

for the case of maximum endurance.

By taking the derivative of Eq. (54) with respect to θ, the variable thrust control will appear linearly and, upon solving, we have

$$\frac{2T}{kpS} = \left[\frac{4+(AC_{D_o}/K)_M}{4+(AC_{D_o}/K)_M + 2\delta^{(n-1)/2n} Mk_c} \right] \frac{(1+A)}{2} \; C_D^* M^2 \tag{57}$$

The bracketed coefficient is the correctional factor to either Eq. (51) or (52). The effect is due to the term $2\delta^{(n-1)/2n} Mk_c$. Because of this term, the thrust is slightly less than the drag,

hence accounting for a slow decrease of the Mach number along the flight path in both the maximum endurance and maximum range cases.

Subsonic and Supersonic Flight Domains

Since η is constant, it is convenient to represent the flight domain in the (M,ω) space. In this space, we plot the relation (54) which represents the trajectory along the sustaining flight arc, with $A = A_R$ for the case of maximum range and $A = A_E$ for the case of maximum endurance.

In this report, two typical military aircraft are used for the numerical application. They are the transport-tanker KC-135A and the fighter F-4C. Their aerodynamic characteristics are given in Appendix A. For both aircrafts, the value k_c is very small and is practically independent of the Mach number. Hence, in the (M,ω) space we have the simplified equations for the trajectory in sustained flight

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} \sqrt{\frac{1 + C_{D_{o_M}}}{3 - K_M}} \tag{58}$$

for the case of maximum range, and

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} \sqrt{\frac{2 + C_{D_{o_M}}}{2 - K_M}} \tag{59}$$

for the case of maximum endurance.

In Appendix A, for each aircraft, the functions C_{D_o} and K are given in terms of the Mach number. The Eqs. (58) and (59) are

plotted in Fig. 3 for the KC-135A and in Fig. 4 for the F-4C.

At low Mach number, the influence of the Mach number on aerodynamic characteristics is negligible and the curves are nearly the parabolas

$$\omega = \frac{C_L^*}{\sqrt{3}} M^2$$
$$\omega = C_L^* M^2 \tag{60}$$

where C_L^* is the asymptotic constant value of C_L^* for low Mach number. Hence, at low Mach number, maximum endurance flight is conducted at $C_L = C_L^*$ while maximum range flight is conducted at $C_L = C_L^*/\sqrt{3}$. At high Mach number, as the logarithmic derivative K_M approaches the value 2 for the case of maximum endurance and the value 3 for the case of maximum range, the curves tend respectively to their vertical asymptotes. From the figures, it is seen that in both cases, optimum Mach number has an upper limit and this upper limit is in the subsonic range. Also for the same $\omega = 2W/kpS$, the Mach number for maximum range is higher than the Mach number for maximum endurance. At low speed, since C_L^* is the same, it is seen from Eqs. (60) that their ratio is $M_R/M_E = (3)^{1/4} = 1.316$. This ratio is smaller for higher ω which corresponds to the case of sustained flight at high altitude and/or high wing loading.

The curves can be used if they are within the flight domain of the aircraft. More specifically, since the lift coefficient C_L is bounded by the maximum lift coefficient $C_{L_{max}}(M)$, which is a

function of the Mach number, in the (M,ω) space, the domain of flight, also called the flight envelope, is bounded by the curve

$$\omega = C_{L_{max}}(M)\, M^2 \tag{61}$$

At low Mach number, $C_{L_{max}}$ is nearly constant, and the curve is approximately a parabola. At high Mach number, $C_{L_{max}}$ depends strongly on the Mach number, and we have two cases:

For subsonic aircraft configuration, when the Mach number is approaching unity, $C_{L_{max}}$ drops drastically and we have the case of Fig. 3. In practice, the domain of flight is further limited by the buffeting effect. At high angle-of-attack, before the lift coefficient reaches its maximum value $C_{L_{max}}$, appreciable flow separation from the wing occurs at a certain lift coefficient $C_{L_B} < C_{L_{max}}$ called the buffeting coefficient. Beyond this lift coefficient, the aircraft begins to shake due to the turbulence associated with flow separation. Like $C_{L_{max}}$, the buffeting lift coefficient C_{L_B} is a function of the Mach number. Hence, if we plot Eq. (61) with $C_{L_{max}}$ replaced by C_{L_B}, we have another curve delimiting the buffeting region.

In any instant of sustained flight, for any prescribed ω, there are 4 Mach numbers of interest namely M_1, M_2, M_E and M_R (Fig. 3). The lower bound Mach number M_1 is the stalling Mach number while the upper bound Mach number M_2 is the critical Mach number due to the subsonic nature of the aircraft aerodynamic configuration. M_1 and M_2 are coincident when the function (61) has its maximum value.

The corresponding Mach number is obtained by writing the condition for the maximum value of the function $M^2 C_{L_{max}}$ which, in terms of the logarithmic derivative, can be written as

$$(C_{L_{max}})_M + 2 = 0 \tag{62}$$

where, as usual, subscript M denotes the logarithmic derivative. The resulting value of ω corresponds to the aerodynamic ceiling.

The Mach number M_R is the Mach number for maximum range and the Mach number M_E is the Mach number for maximum endurance. They are obtained by solving the Eq. (54) with $A = A_R$ and $A = A_E$ respectively.

For supersonic aircraft configuration, the function $C_{L_{max}}(M)$ decreases when the Mach number approaches unity and then tends to a near constant value, lower than the constant value for low subsonic regime. Hence, in the supersonic regime, the function (61) is again near parabolic as seen in Fig. 4. In this case, while the lower bound Mach number M_1 is still the stalling Mach number, the upper bound Mach number M_2 is either the critical Mach number due to the effect of compressible fluid or the maximum Mach number that maximum thrust can deliver in level flight. This Mach number is obtained by solving the equation

$$\eta\tau_{max} = M^2\left(C_{D_o} + \frac{k\omega^2}{M^4}\right) \tag{63}$$

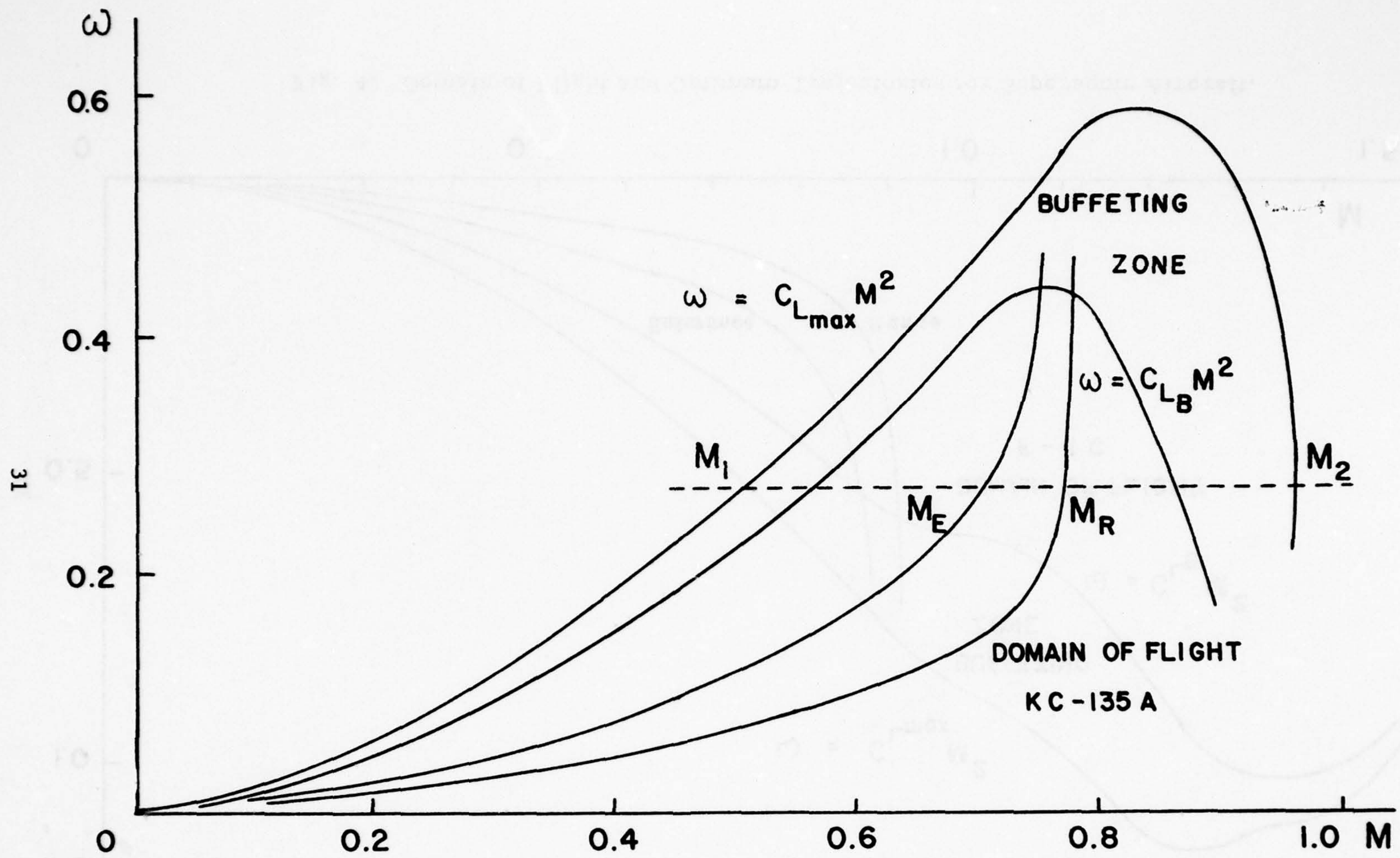

Fig. 3. Domain of Flight and Optimum Trajectories for Subsonic Aircraft.

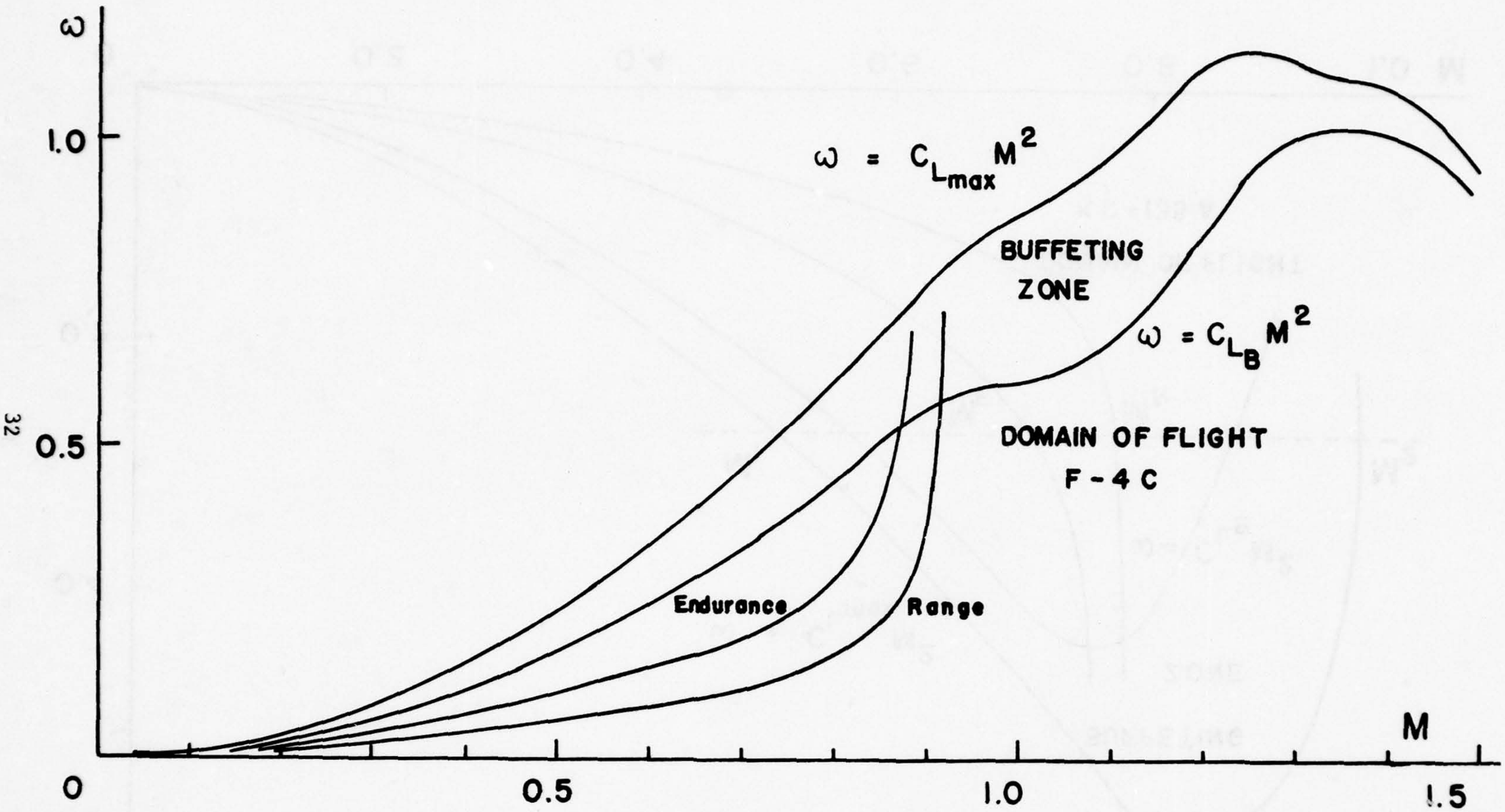

Fig. 4. Domain of Flight and Optimum Trajectories for Supersonic Aircraft.

Optimum Switching

The general problem of constant-altitude flight is formulated as follows. At a given constant altitude, the initial and the final Mach numbers M_i and M_f are prescribed. For a given initial weight W_i and a final weight W_f, find the optimum aerodynamics and engine controls to maximize the range or the endurance.

To solve the problem, we use the dimensionless wing loading ω to replace the weight. For any given aircraft aerodynamic configuration and engine characteristic we plot the optimum relation (54) between ω and M in the (M,ω) space. (Fig. 5). This curve represents the optimum subarc for variable thrust sustaining flight. In general, a complete optimum trajectory connecting the initial point (M_i,ω_i) and the final point (M_f,ω_f), inside the permissible domain of flight, includes an initial arc which can be a null thrust arc, $\tau = 0$, or a maximum thrust arc $\tau = \tau_{max}$ joining the initial point and the variable thrust arc, and also a final arc with $\tau = 0$, or $\tau = \tau_{max}$ leading to the final point.

The figure represents the case where the trajectory starts with a maximum thrust arc to increase the Mach number to a value such that the optimum relation (54) is satisfied. From that point, the optimum Mach number is maintained such that the relation is identically satisfied. If the final Mach number is free or at a value to the left of the curve, variable thrust is maintained until ω reaches ω_f. At that point, the thrust is reduced to zero and the aircraft coast flies to the final value M_f or to the stalling Mach

number, providing either the maximum range, or the maximum endurance depending on the expression used for the function A. Along the sustaining arc the optimum lift coefficient is given by either Eq. (45) or (46), and the optimum modulated thrust is given by Eq. (57). Along the initial maximum thrust arc, $\tau = \tau_{max}$ and along the final coasting arc $\tau = 0$, while the lift coefficient is adjusted such that the equilibrium condition (26) is satisfied. It appears now that the maximum thrust arc and the null thrust arc are short arcs necessary to connect the boundary Mach number to or from the optimum Mach number along the sustaining arc.

The junction of two arcs of different types is called a switching. The direction of optimum switching can be decided upon by the following rule (Ref. 4). Let

$$\begin{aligned} H_1 &= (H)_{\tau = 0} \\ H_2 &= (H)_{\tau = \tau_{max}} \end{aligned} \tag{64}$$

and define the switching function Θ as

$$\Theta = H_2 - H_1 \tag{65}$$

Then, the switching is from $\tau = 0$ to $\tau = \tau_{max}$ if and only if at the switching point

$$\frac{d}{d\theta} \Theta > 0 \tag{66}$$

where in taking the derivatives, the state and adjoint equations

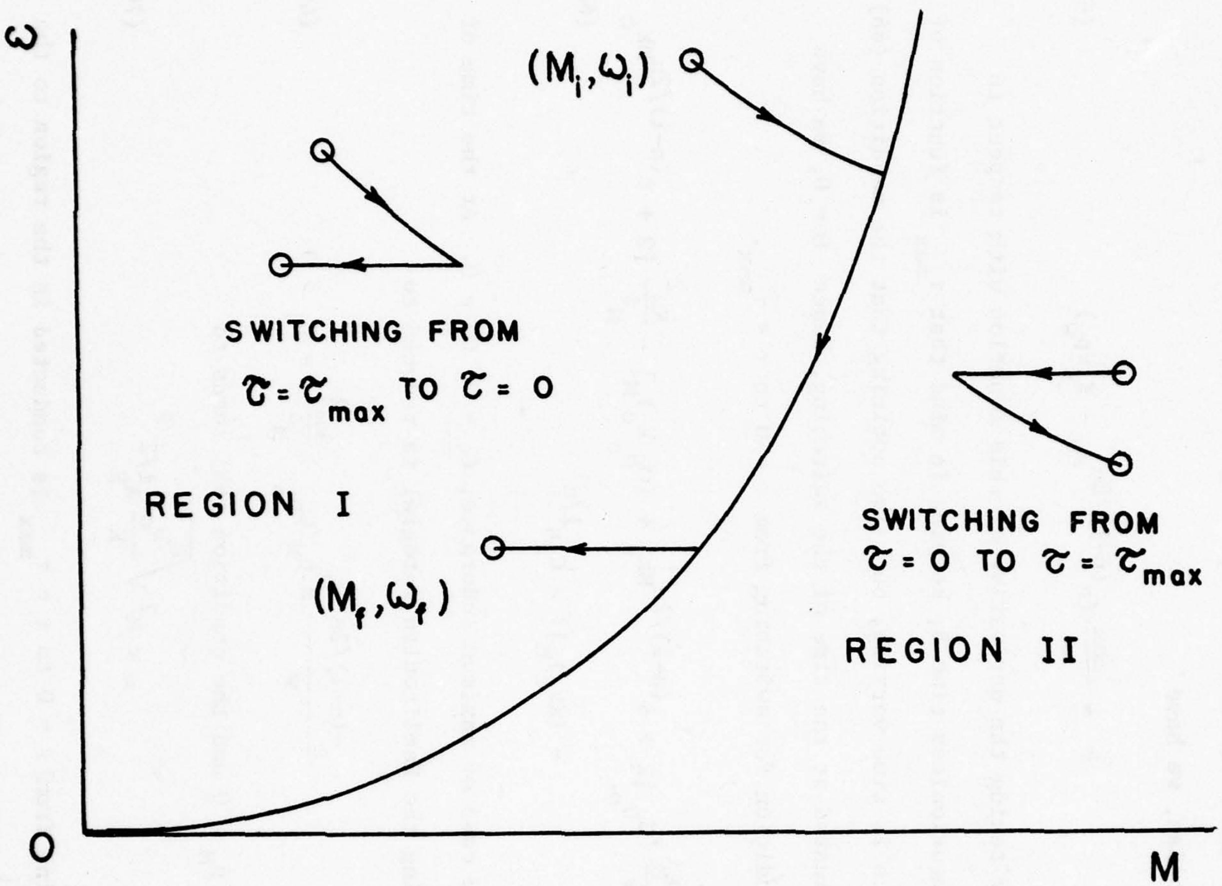

Fig. 5. Optimum Switchings.

are used with $\tau = 0$. If the inequality reverses, then the switching is from $\tau = \tau_{max}$ to $\tau = 0$.

In applying the theorem, using the expression (29) for the Hamiltonian, we have

$$\Theta = \frac{\tau_{max}}{w}(\eta^{(n-1)/2n} p_M - K_c w p_w) \tag{67}$$

By taking the derivative of this equation with respect to the dimensionless time θ, keeping in mind that τ_{max} is function of M, hence is time varying, but also noticing that the condition (66) is evaluated at the time of the switching, hence $\Theta = 0$, we have the condition for switching from $\tau = 0$ to $\tau = \tau_{max}$.

$$\frac{Mp_M}{w}\{C_{D_o}[2 + \delta^{(n-1)/2n}Mk_c + (C_{D_o}k_c)_M] - \frac{K\omega^2}{M^4}[2 + \delta^{(n-1)/2n}Mk_c - (Kk_c)_M]\} > C_2\eta^{1/n} \tag{68}$$

For the case of maximum endurance, $C_2 = 0$, $C_1 > 0$. At the time of switching the Hamiltonian integral is reduced to

$$\frac{\eta^{-(n+1)/2n}}{w} M^2 p_M (C_{D_o} + \frac{k\omega^2}{M^4}) = C_1 > 0 \tag{69}$$

Hence, $p_M > 0$ and the condition (68) leads to

$$\omega < M^2 \sqrt{\frac{C_{D_o}}{K}}\, A_E^{1/2} \tag{70}$$

Switching from $\tau = 0$ to $\tau = \tau_{max}$ is conducted in the region to the right of the sustaining trajectory and from $\tau = \tau_{max}$ to $\tau = 0$ in the region to the left of the curve.

For the case of maximum range, $C_1 = 0$, $C_2 > 0$. The Hamiltonian integral is reduced to

$$\frac{Mp_M}{w}\left(C_{D_o} + \frac{K\omega^2}{M^4}\right) = C_2 n^{1/n} > 0 \tag{71}$$

Hence, we also have $P_M > 0$, and combining the Eqs. (68) and (71) we have the condition for a switching from $\tau = 0$ to $\tau = \tau_{max}$

$$\omega < M^2 \sqrt{\frac{C_{D_o}}{K}}\, A_R^{1/2} \tag{72}$$

We have the same conclusions for the different regions of optimum switching (Fig. 5).

The sustaining arc, as given by Eq. (54) divides the admissible flight domain in the (M,ω) space into two regions I and II. In the region I, the switching is from $\tau = \tau_{max}$ to $\tau = 0$, while in the region II the switching is from $\tau = 0$ to $\tau = \tau_{max}$. Let

N: coast arc, null thrust, $\tau = 0$

M: maximum thrust arc, $\tau = \tau_{max}$

S: sustaining arc, τ = variable

If the initial point is in region I, the trajectory starts with maximum thrust arc, while it starts with a coast arc if the initial point is in region II. Likewise, if the final point is in the region I, the trajectory terminates with a coast arc while it terminates with a maximum thrust arc if the final point is in the region II. If the final Mach number is free, at the final time

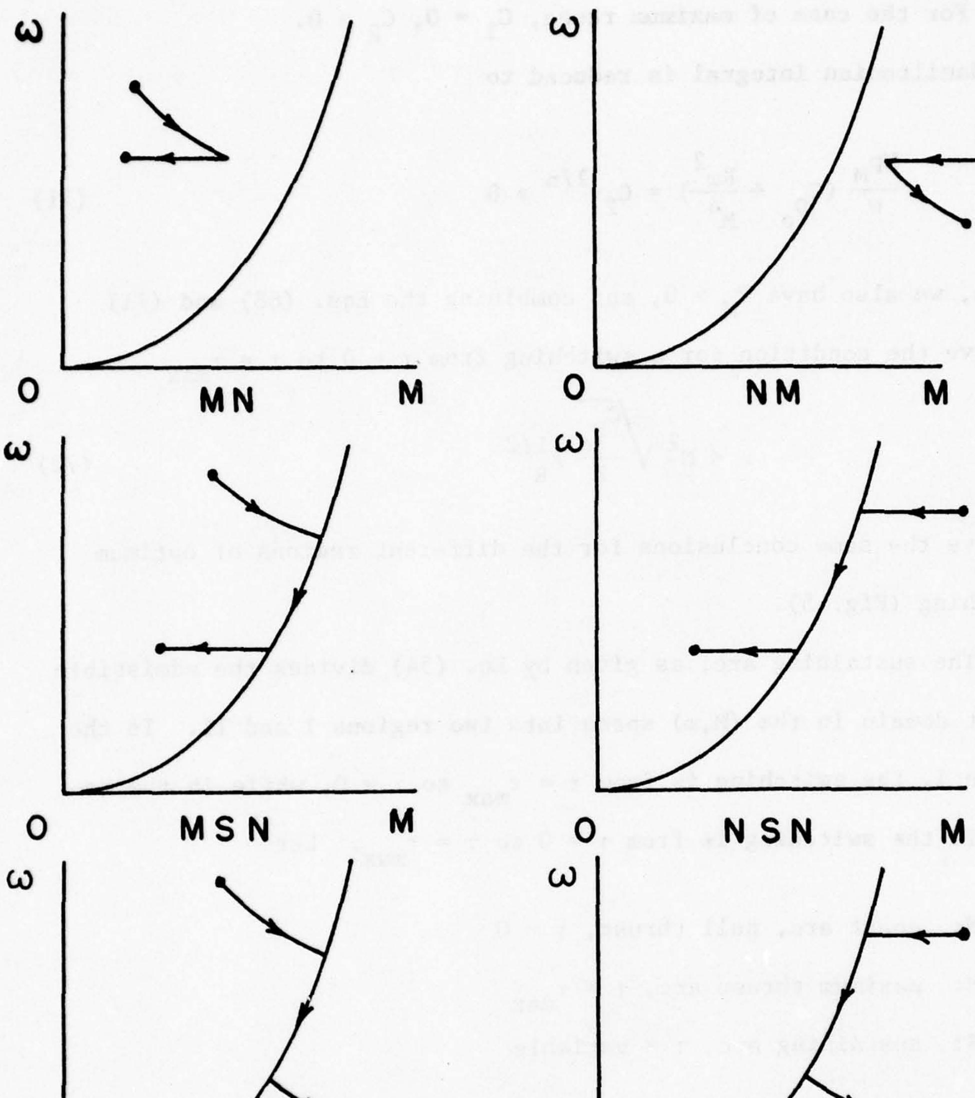

Fig. 6. Six Types of Optimum Trajectories.

$P_{M_f} = 0$, and by the definition (32), Θ is necessarily negative. Hence, the optimum trajectory terminates with a coasting arc.

With these considerations we have six types of optimum trajectories as shown in Fig. 6. In general, for both the maximum range problem and the maximum endurance problem, the weight of the fuel $W_i - W_f$ is large enough such that the main portion of the optimum trajectory is a sustaining arc. We have ruled out the case of pure coasting flight $\Delta W = 0$, and the case of pure maximum thrust flight since they are special cases with the sustaining arc vanishing. Using the theory in Ref. 4, it can be proved rigorously that there are no more than two switchings and Fig. 6 summarizes all possible optimum trajectories.

SECTION VI

CRUISE PERFORMANCE

Again, we consider the problem of maximum range and the problem of maximum endurance. While in the previous section the flight is constrained to perform at constant altitude, this restriction is removed in this section. Obviously, performance is improved without constraint on phase space. This is one of the most attractive aspects of cruise performance.

Generally speaking, optimum cruise is performed at very small flight path angle. Hence, we can take approximately $\cos\gamma \simeq 1$ and $\sin\gamma \simeq \gamma$ and write the general equations in the form

$$\begin{aligned}
\frac{dx}{d\theta} &= \eta^{(1-n)/2n} M \\
\frac{d\eta}{d\theta} &= K_B \eta^{(n+1)/2n} M\gamma \\
\frac{dM}{d\theta} &= \frac{\eta^{(n-1)/2n}}{w}\left[\tau - \eta^{-1}M^2(C_{D_o} + KC_L^2)\right] \\
\frac{d\gamma}{d\theta} &= \frac{\eta^{(n-1)/2n}}{wM}\left(\eta^{-1}M^2 C_L - w\right) \\
\frac{dw}{d\theta} &= -K_c\tau \\
\frac{d\theta}{d\theta} &= 1
\end{aligned} \tag{73}$$

where, in the equation for M, we have neglected the small component of the gravity $w \sin\gamma$ as compared to the drag force.

The optimum aerodynamics and engine controls for both problems will be derived by two different, but closely related, analytical

approaches. The first one will be given in this section.

Equilibrium Cruise

For cruise performance, the flight path angle not only remains small but varies slowly along the cruise portion of the trajectory. Hence, we can take $d\gamma/d\theta = 0$, and obtain the equilibrium cruise condition

$$C_L = \frac{\eta w}{M^2} \tag{74}$$

This has the result of replacing the lift coefficient C_L by the altitude η as one of the controls, the other being the dimensionless thrust τ.

On the other hand, since the altitude varies slowly, we can remove the equation for η from consideration. Both assumptions are justified by the fact that for most aircraft in use, over a range of several hundred nautical miles the gain in altitude in optimum cruise is only a few thousand feet. Hence, both the variations in η and in γ are negligibly small.

With these assumptions, the system (73) is reduced to

$$\begin{aligned}
\frac{dx}{d\theta} &= \eta^{(1-n)2n} M \\
\frac{dM}{d\theta} &= \frac{\eta^{(n-1)/2n}}{w} [\tau - \eta^{-1} M^2 (C_{D_o} + \frac{K\eta^2 w^2}{M^4})] \\
\frac{dw}{d\theta} &= - K_c \tau \\
\frac{d\theta}{d\theta} &= 1
\end{aligned} \tag{75}$$

which is identical to system (27) for level flight. Nevertheless, the main difference here is that η is not a constant parameter, but in this formulation, it is considered a control variable which is varying slowly such that approximately $d\eta/d\theta = 0$. It should be noted that the assumptions are introduced solely to obtain an analytical solution for the optimum aerodynamics and engine controls. Once the solution for the controls has been obtained, the elements of the trajectory, and in particular the altitude η and the flight path angle γ are obtained by integrating the full set of equations, the Eqs. (73), or better, the exact system (19) with the equation in z replaced by the equation (22) in η.

The analysis is the same as in the case of level flight. In particular, along the main portion of sustaining flight we have the relations (35) and (37) rewritten here for convenience

$$C_2\eta^{1/n}M = \frac{M^2 p_M}{w}\{C_{D_o}[2 + \eta^{(1-n)2n}MK_c + (C_{D_o}K_c)_M] - \frac{K\eta^2 w^2}{M^4}[2 + \eta^{(1-n)/2n}MK_c - (KK_c)_M]\} \tag{76}$$

$$C_1\eta^{(n+1)/2n} + C_2\eta^{1/n}M = \frac{M^2 p_M}{w}(C_{D_o} + \frac{K\eta^2 w^2}{M^4}) \tag{77}$$

Again, we consider the expression for the Hamiltonian

$$H = C_1 + C_2\eta^{(1-n)/2n}M - \frac{\eta^{-(n+1)/2n}}{w}M^2 p_M(C_{D_o} + \frac{K\eta^2 w^2}{M^4}) + \frac{\tau}{w}[\eta^{(n-1)/2n}p_M - K_c w p_w]$$

With respect to η, considered as a control, H is maximized when $dH/d\eta = 0$.

We have

$$(1-n)C_2\eta^{1/n}M + \frac{M^2 p_M}{w}[(n+1)C_{D_o} - (3n-1)\frac{K\eta^2 w^2}{M^4}] + (n-1)\frac{\eta\tau}{w}p_M = 0$$

Since the Mach number varies slowly the thrust is nearly equal to the drag and we have

$$\eta\tau = M^2(C_{D_o} + KC_L^2)$$

Substituting into the equation above, we have the relation

$$(n-1)C_2\eta^{1/n}M = \frac{2nM^2 p_M}{w}(C_{D_o} - \frac{K\eta^2 w^2}{M^4}) \qquad (78)$$

By eliminating $M^2 p_M/w$ among the Eqs. (76), (77) and (78) we have two relationships for the weight w, the altitude η and the Mach number M along the sustaining arc.

The following particular cases are of interest:

Maximum Range

In the problem of maximum range with free final time, we have $C_1 = 0$. Then the Eqs. (76), (77) and (78) become

$$C_2\eta^{1/n}M = \frac{M^2 p_M}{w}\{C_{D_o}[2 + \eta^{(1-n)/2n}MK_c + (C_{D_o}K_c)_M] - \frac{K\omega^2}{M^4}[2 + \eta^{(1-n)/2n}MK_c - (KK_c)_M]\}$$

$$C_2\eta^{1/n}M = \frac{M^2 p_M}{w}(C_{D_o} + \frac{K\omega^2}{M^4})$$

$$(n-1)C_2 n^{1/n}M = \frac{2nM^2 p_M}{w}\left(C_{D_o} - \frac{K\omega^2}{M^4}\right)$$

Eliminating C_2 between the last two equations, we have

$$\omega = \sqrt{\frac{n+1}{3n-1}}\sqrt{\frac{C_{D_o}}{K}}\,M^2 \tag{79}$$

On the other hand, the elimination of C_2 between the first two equations yields

$$\omega = M^2\sqrt{\frac{C_{D_o}}{K}}\;A_R^{1/2} \tag{80}$$

where

$$A_R = \frac{1 + \delta^{(n-1)/2n}Mk_c + (C_{D_o}k_c)_M}{3 + \delta^{(n-1)/2n}Mk_c - (Kk_c)_M} \tag{81}$$

where $\delta = p/p_*$ is the pressure ratio. The equilibrium condition (74) and the optimum relations (79) and (80) provide the optimum lift control law, the optimum Mach number and the optimum cruise altitude for the problem of maximum range.

Combining the Eqs. (74) and (79), we have the optimum lift coefficient as given in Ref. 1

$$C_L = \sqrt{\frac{n+1}{3n-1}}\,C_L^* \tag{82}$$

In terms of the lift-to-drag ratio, we have

$$\frac{C_L}{C_D} = \frac{\sqrt{(n+1)(3n-1)}}{2n}\,E_{max} \tag{83}$$

Thus, in the stratosphere where n = 1, the flight is at maximum lift-to-drag ratio, while in the troposphere where n = 1.235

$$\frac{C_L}{C_D} = 0.99546\ E_{max} \qquad (84)$$

In terms of the aerodynamic control, optimum cruise for maximum range is quite different than in the case of level flight. This can be interpreted as while holding down the aircraft to maintain constant altitude, performance is diminished, but if the restriction is removed, the aircraft will climb to the optimum altitude and cruise in the optimum balance between lift and drag.

Eliminating ω between the Eqs. (79) and (80), we have the equation for evaluating the optimum Mach number

$$A_R = (n+1)/(3n-1) \qquad (85)$$

Explicitly, we have

$$2(n-1)\delta^{(n-1)/2n} Mk_c + (n-1)(Kk_c)_M + (3n-1)(C_{D_o}k_c)_M = 4 \qquad (86)$$

If the flight is performed in the stratosphere, n=1, and we have

$$\left(\frac{k_c}{E_{max}}\right)_M = 1 \qquad (87)$$

It is seen that, the optimum Mach number is function of the specific fuel consumption coefficient k_c defined by Eq. (41) and the maximum lift-to-drag ratio. Hence, while for flight in the

troposphere, the optimum Mach number, as given by Eq. (86), slightly depends on the altitude through the pressure ratio δ besides its aerodynamics and engine characteristics dependence, for flight in the stratosphere, to the accuracy of the assumptions introduced above, the optimum Mach number as given by Eq. (87) is a constant.

For flight in the stratosphere we have the following pertinent remarks:

1. The function

$$\frac{k_c}{E_{max}} = f(M) \tag{88}$$

is the similarity function for cruise performance. That is to say if two aircrafts have the same function f(M), they will fly at the same Mach number for maximum range.

2. Since the Mach number is constant, C_{D_o} and K are constant and by Eq. (79) $\omega = \eta w$ is constant. Since w decreases, η increases and hence the altitude increases.

3. As mentioned above, the optimum law is only approximate. Using this approximate law in the exact equations for integration the resulting product ηw is not a constant but is slowly varying.

4. With the approximation ω = constant, we write

$$\frac{\rho_i}{\rho_f} = \frac{p_i}{p_f} = \frac{W_i}{W_f} \tag{89}$$

Hence, if an exponential atmosphere is used, the gain in altitude over the whole sustaining arc is

$$Z_f - Z_i = \frac{1}{\beta} \text{Log} \frac{W_i}{W_f} \tag{90}$$

and is of the order of a few thousand feet. Over a range of a few hundred nautical miles and considering the trajectory as nearly a straight line, the flight path angle, γ, expressed in radians is negligibly small.

A complete optimum trajectory will include a climb to a certain optimum altitude where the variable thrust phase will begin. As in the case of level flight, the optimum trajectory ends with a climb to the final prescribed altitude with the final prescribed Mach number, or a glide to the final altitude. This complete problem is difficult to solve analytically.

Approximately, we can consider a suboptimum climb to an altitude near the tropopause with a resulting weight W_i to be used as the initial weight for the sustaining phase. Then the optimum cruise altitude and the optimum Mach number are obtained by solving the two equations (79) and (86) for flight in the troposphere. For flight in the stratosphere, the Mach number is obtained from Eq. (87) and then the altitude is given directly by Eq. (79) with $\omega = \eta w_i$.

The equation for the optimum Mach number can be solved only for given aerodynamics and engine characteristics as will be carried out later in this section. Nevertheless, following Miller and Koch (Ref. 1), we can generate some useful graphs for use with any aircraft. In this purpose, we rewrite Eq. (79) as

$$\frac{W}{kpSC_{D_o}} = \frac{\sqrt{n+1}\, M^2}{2\sqrt{(3n-1)KC_{D_o}}} \tag{91}$$

or

$$\sqrt{\frac{3n-1}{n+1}}\left(\frac{W}{kp_*SC_{D_o}}\right)\frac{1}{\delta} = E_{max}M^2 \tag{92}$$

where $\delta = p/p_*$ is the pressure ratio. Then, we consider the graphs of the two functions

$$y = E_{max}\, M^2 \tag{93}$$

and

$$y = \frac{\Omega}{\delta} \tag{94}$$

where the parameter Ω is defined as

$$\Omega = \sqrt{\frac{3n-1}{n+1}}\left(\frac{W}{kp_*SC_{D_o}}\right) \tag{95}$$

Fig. 7 presents the plot of Eq. (93) for different values of E_{max}, and the plot of Eq. (94) for different values of the modified wing loading Ω. For convenience, in this plot, the pressure ratio δ has been expressed in terms of the altitude, and the ordinate is plotted in terms of Log y.

As application, consider a flight in the stratosphere. For a certain aerodynamic configuration and engine characteristics, Eq. (87) provides the constant optimum Mach number, say M = 0.85. Then C_{D_o} (M) and K(M) can be evaluated and consequently also the value

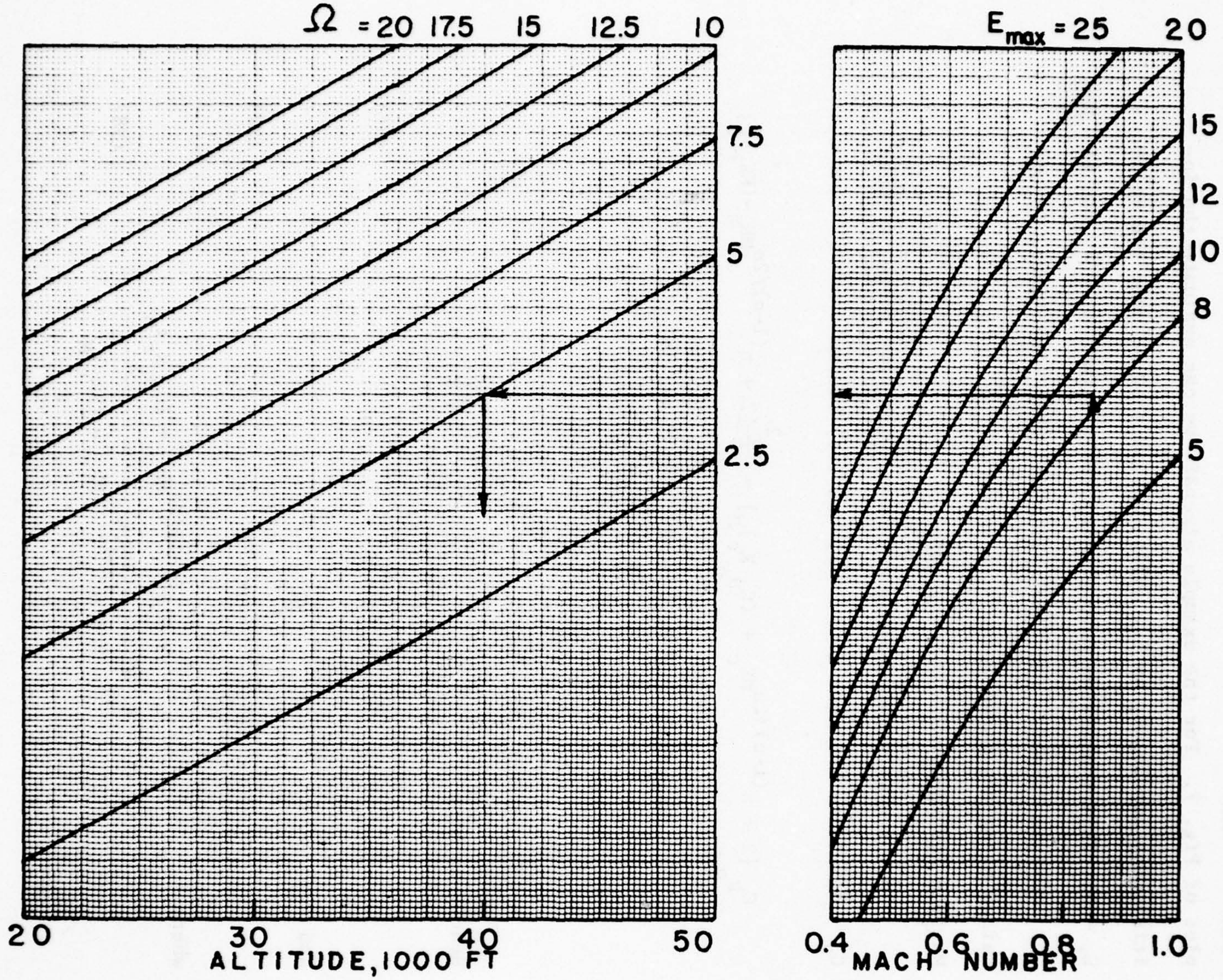

Fig. 7. Optimum Cruise Altitude.

E_{max}, say $E_{max} = 8.4$. This provides the operating point on the right plot of Fig. 7. For the example aircraft we can evaluate the initial wing loading Ω_i by Eq. (95). Assume that we have the value $\Omega_i = 5$. This translates into the initial altitude $Z_i = 40{,}000$ ft for cruising flight. The final altitude, which is higher, is obtained along the curve Ω_f.

Maximum Endurance

In the problem of maximum endurance with free range, we have $C_2 = 0$. Then, the Eqs. (76), (77) and (78) become

$$C_{D_o}[2 + \eta^{(1-n)/2n}MK_c + (C_{D_o}K_c)_M] = \frac{K\omega^2}{M^4}[2 + \eta^{(1-n)/2n}MK_c - (KK_c)_M]$$

$$\eta^{(n+1)/2n}C_1 = \frac{M^2 p_M}{w}(C_{D_o} + K\frac{\omega^2}{M^4})$$

$$C_{D_o} = \frac{K\omega^2}{M^4}$$

Hence, we have

$$\omega = M^2\sqrt{\frac{C_{D_o}}{K}} \qquad (96)$$

and

$$\omega = M^2\sqrt{\frac{C_{D_o}}{K}}\,A_E^{1/2} \qquad (97)$$

where

$$A_E = \frac{2 + \delta^{(n-1)/2n}Mk_c + (C_{D_o}k_c)_M}{2 + \delta^{(n-1)/2n}Mk_c - (Kk_c)_M} \qquad (98)$$

In principle, the equilibrium condition (74) and the optimum relation (96) and (97) constitute a system of three equations for evaluating the optimum lift coefficient, the optimum Mach number and the optimum cruise altitude for the problem of maximum endurance. We will see later that because the effect of the altitude on the endurance is of the second order a certain indetermination occurs when we evaluate the optimum altitude.

By combining the Eqs. (74) and (96) we have the optimum lift coefficient

$$C_L = \sqrt{\frac{C_{D_o}}{K}} = C_L^* \tag{99}$$

This gives

$$\frac{C_L}{C_D} = \frac{1}{2\sqrt{KC_{D_o}}} = E_{max} \tag{100}$$

Thus, flight for maximum endurance is at maximum lift-to-drag ratio to the accuracy provided by the assumption of equilibrium cruise. In terms of the aerodynamic control, optimum cruise for maximum endurance is nearly the same as in the case of level flight.

Eliminating ω between the Eqs. (96) and (97) we have the equation for evaluating the optimum Mach number

$$A_E = 1 \tag{101}$$

Explicitly, we have

$$\left(\frac{k_c}{E_{max}}\right)_M = 0 \tag{102}$$

It is seen from this equation that optimum Mach number is constant. We have the same remarks as given above for the case of maximum range in the stratosphere. Once, the Mach number has been obtained from Eq. (102), the optimum flight altitude is given by Eq. (96). We can write this equation

$$\frac{1}{\delta}\left(\frac{W}{kp_{*}SC_{D_o}}\right) = E_{max}M^2 \tag{103}$$

As before, with the case of maximum range, we consider the plots

$$\begin{aligned} y &= E_{max}M^2 \\ y &= \frac{\Omega}{\delta} \end{aligned} \tag{104}$$

where

$$\Omega = \frac{W}{kp_{*}SC_{D_o}} \tag{105}$$

Therefore, Fig. 7 can be used with this new definition of Ω which is the same as the case of maximum range in the stratosphere.

This doesn't mean that, for a given initial weight, flight for maximum range and flight for maximum endurance are performed at the same altitude. In general, flight for maximum range is at higher Mach number and higher altitude than flight for maximum endurance.

While for the case of maximum range, once the aerodynamics and engine characteristics are given, one can always evaluate the optimum Mach number and the optimum altitude, there is a certain indetermination in the case of maximum endurance as has been observed by

Miller and Koch (Ref. 1).

To explain this ambiguity, let us consider the optimization problem from the point of view of point performance. The usual assumption is that of unaccelerated flight. Hence, from the equation in M in the system (73)

$$\eta\tau = M^2(C_{D_o} + KC_L^2) = M^2 C_D \tag{106}$$

Using the equilibrium condition (74) in this equation for evaluating τ and then substituting into the equation for w, we have

$$\frac{d}{d\theta}(\text{Log } w) = -\frac{K_c}{(C_L/C_D)} \tag{107}$$

At each instant, to maximize instantaneously the endurance, we are led to minimizing the mass flow, that is the right hand side of Eq. (107). This clearly shows the trade-off between the specific fuel consumption coefficient K_c and the lift-to-drag ratio C_L/C_D as has been mentioned earlier. Since this coupling is weak, we see that maximum lift-to-drag ratio must be used. Then the next step is to find the minimum of K_c/E_{max}, considered as a function of the Mach number, a condition expressed by Eq. (102). Unfortunately, for a typical aircraft, E_{max} as a function of the Mach number is a maximum at low Mach number where the effect of compressibility is negligible. That is to say Eq. (102) is identically satisfied for a whole range of low Mach numbers. The optimum Mach number is undetermined and subsequently, from Eq. (96) we have a whole range of altitude for optimum cruise.

The conclusion is that the effect of the altitude on maximum endurance is not detected to the first order of approximation. The only requirement in aerodynamic control is that the flight be performed at maximum lift-to-drag ratio. For each selected altitude, there will be an optimum Mach number as given by Eq. (96).

To detect the second order effect of the altitude, singular variable thrust control is required. As has been shown in the preceding section the varying thrust magnitude is given by Eq. (57) which contains the influence of the altitude through the pressure ratio δ. Since maximum endurance cruise is performed at low Mach number, we can neglect the effect of compressibility and, from Eq. (98) take $A_E = 1$, and then have the expression (57) in the form

$$\frac{T}{kpS} = \left(\frac{2}{2 + \delta^{(n-1)/2n} Mk_c}\right) C_{D_o} M^2 \tag{108}$$

Using the optimum condition (96) we have

$$\frac{\tau}{w} = \frac{1}{E_{max}} \left(\frac{2}{2 + \delta^{(n-1)/2n} Mk_c}\right) \tag{109}$$

Therefore, we can write the equation in w in system (73)

$$\frac{d}{d\theta} (\text{Log } w) = - \frac{k_c}{E_{max}} \left(\frac{2}{2 + \delta^{(n-1)/2n} Mk_c}\right) \tag{110}$$

As compared to Eq. (107) we now have the effect of the altitude through the pressure δ. To minimize the mass flow, we are led to maximizing the denominator on the right hand side of Eq. (110). When E_{max} is constant, that is when the optimum Mach number is in

the low subsonic range, we maximize the term $\delta^{(n-1)/2n} Mk_c$ which, because of the optimum relation (96), can be expressed as

$$\delta^{(n-1)/2n} Mk_c = \left(\frac{2w}{kp_* SC_L^*}\right)^{1/2} \frac{k_c}{\delta^{1/2n}} \tag{111}$$

Maximizing this term is equivalent to taking the smallest value of the pressure ratio δ. Hence the optimum altitude is the highest one possible provided that the corresponding Mach number as obtained from Eq. (96) is less than the critical Mach number M_1 beyond which the effect of compressible fluid is being felt. This effect of the altitude, at low subsonic range, is small since the term $(2W/kp_* SC_L^*)^{1/2}$ is of the order of unity while a typical value of k_c is 8.7749×10^{-3} for a transport aircraft (KC-135A) and 8.9838×10^{-3} for a typical fighter aircraft (F-4C). Hence $k_c/\delta^{1/2n}$ varies from 0.547 k_c at sea level to k_c at the tropopause and the resulting improvement in fuel saving is small. On the other hand, as the selected cruise altitude is further increased, the optimum Mach number is greater than M_1, the effect of compressible fluid drastically decreases E_{max} and the right hand side of Eq. (110) increases in absolute value with the altitude. Hence, the mass flow increases and the endurance decreases.

In summary, the endurance increases slightly with the altitude, until a certain critical altitude is reached. This altitude is obtained from Eq. (96) by using $M = M_1$ where M_1 is the Mach number beyond which the effect of compressible fluid is no longer negligible. Beyond this critical altitude, endurance decreases as the altitude

increases.

Applications

As applications of the analysis we shall consider two typical military aircrafts, the KC-135A and the F-4C. For each aircraft, drag polars for a range of Mach numbers of interest have been obtained from a correlation of wind tunnel tests and flight test results. For each Mach number, the point of maximum lift-to-drag ratio on the polar is identified. If we require that the theoretical parabolic drag polar

$$C_D = C_{D_o} + K C_L^2 \tag{112}$$

matches the same values for C_L^* and E_{max}, then

$$C_{D_o} = \frac{C_D^*}{2} \tag{113}$$

and

$$K = \frac{C_D^*}{2C_L^{*2}} \tag{114}$$

For the KC-135A aircraft, C_{D_o} and K remain nearly constant for M < 0.7 and then increase with the Mach number. The slopes of the two curves are positive and become large as the Mach number approaches unity. The variations are shown in Figs. 8 and 9 with the circles representing the numerical data points.

It is always possible to model the functions C_{D_o} and K as polynomials in M. The inconvenience of this mode of representation is that, in the pertinent formulas we have derived, the derivatives

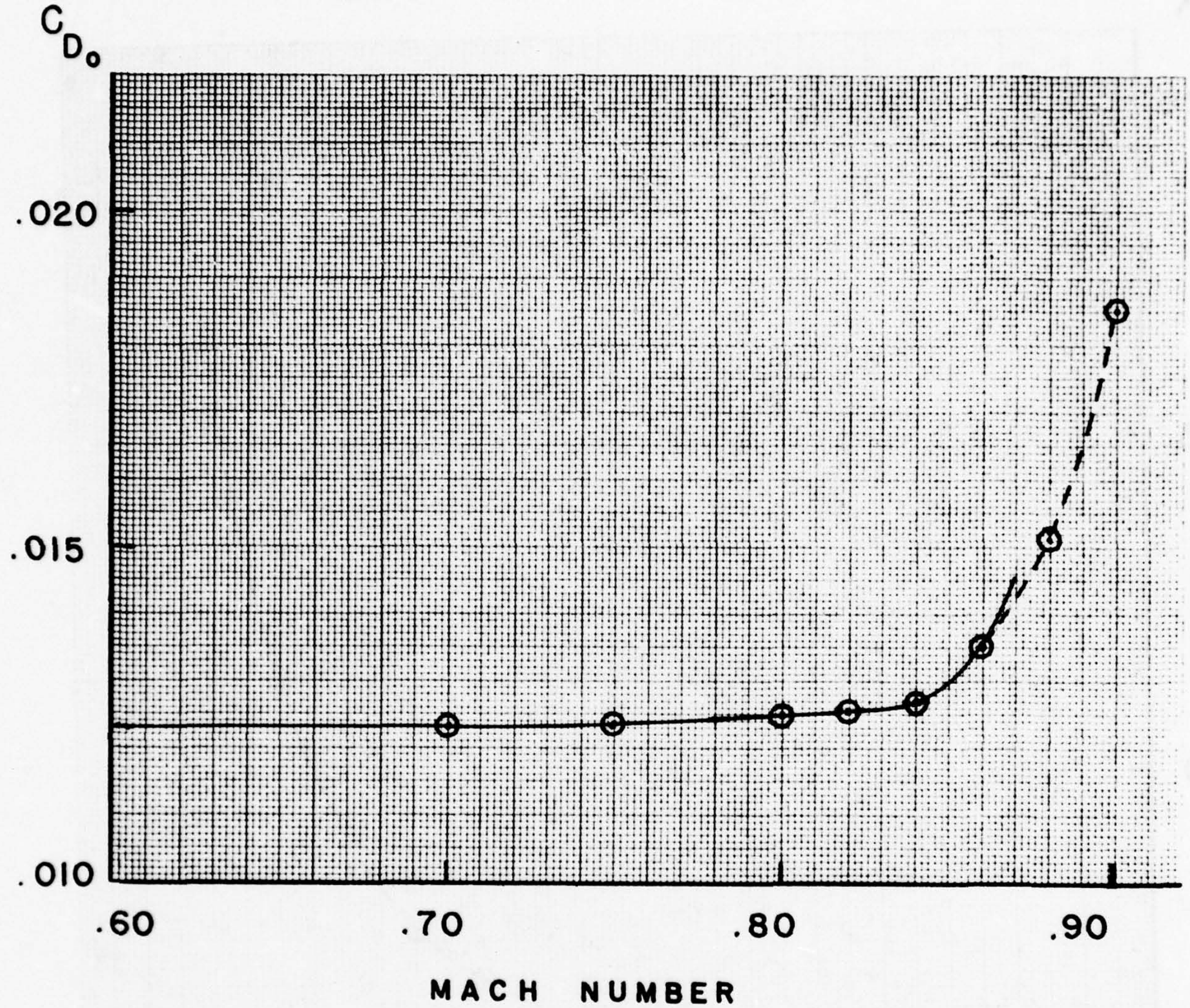

Fig. 8. Variation of C_{D_o} for a Typical Transport Aircraft (KC-135A).

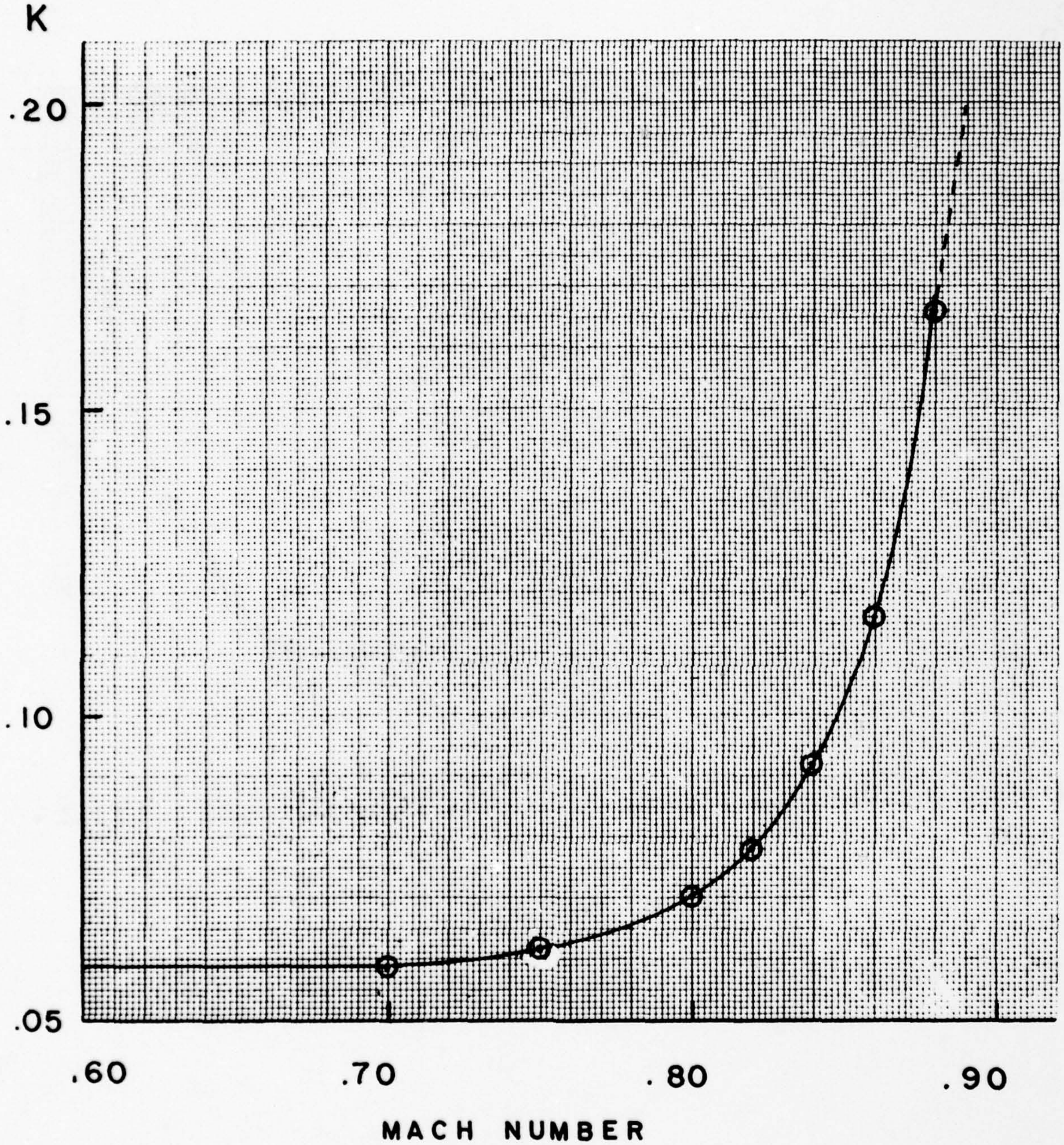

Fig. 9. Variation of K for a Typical Transport Aircraft (KC-135A).

dC_{D_o}/dM and dK/dM also need to be represented accurately. These derivatives are positive and increase with the Mach number. Polynomial representation of C_{D_o} and K may give negative slopes within certain small intervals. Let M_1 be the Mach number beyond which we have the effect of compressibility and C_{D_1} and K_1 be the corresponding values for C_{D_o} and K. Then, for the type of curves such as represented by C_{D_o} and K in Figs. 8 and 9, the following representations give accurately the functions and their first derivatives for $M \geq M_1$

$$C_{D_o} = C_{D_1} + (M-M_1) \sum_{i=1}^{n} \frac{a_i}{(b_i-M)^{c_i}} \tag{115}$$

$$K = K_1 + (M-M_1) \sum_{i=1}^{n} \frac{di}{(e_i-M)^{f_i}} \tag{116}$$

The rationale of this mode of representation is that first, when $M = M_1$, $C_{D_o} = C_{D_1}$, $K = K_1$. Then next when M approaches the smallest of the b_i and e_i, C_{D_o} and K increase rapidly provided that c_i and f_i are positive. The number of data points that can be used to fit the curve C_{D_o} or K, besides the initial point $M = M_1$, is 3n. For example, in the case of the KC-135A aircraft, besides the point $M = 0.70$, we take n=1, and choose the points M = 0.80, 0.84 and 0.86. Then, each of the two equations (115) and (116) provides three equations for evaluating three constant unknowns. The results are

$$C_{D_o} = 0.0123 + \frac{2.29 \times 10^{-4} (M-0.7)}{(0.8665 - M)^{0.69339}} \tag{117}$$

$$K = 0.05864 + \frac{7.30 \times 10^{-4} (M-0.7)}{(0.9742 - M)^{2.884012}} \tag{118}$$

If the last point is selected at $M = 0.90$, the general behavior of the two curves for C_{D_o} and K are correct up to that value of the Mach number but the accuracy in the low Mach number range will decrease unless we take more data points by using n=2.

As shown in Figures 8 and 9, with n=1, for the KC-135A, in the range of Mach number of interest between 0.70 and 0.86 the Eqs. (117) and (118) generate accurate numerical data for C_{D_o} and K and their derivatives. For $M < 0.7$ the values C_{D_1} and K_1 are used for C_{D_o} and K. Figure 10 plots the function E_{max} versus M with excellent agreement with the data from flight tests. It is seen that the maximum of E_{max} occurs when $M < 0.70$. Optimum Mach number for maximum endurance is selected in this range. The optimum Mach number for maximum range is given by Eq. (87) for flight in the stratosphere. Since k_c, as a function of the Mach number is nearly constant, the condition is the same as the condition for the maximum of the function $M\,E_{max}$ which is plotted in Fig. 11. From this figure it is seen that optimum Mach number for maximum range is $M = 0.77$. Numerical solution can be obtained by solving the equation

$$\frac{d}{dM}\left(\frac{M}{\sqrt{KC_{D_o}}}\right) = 0 \tag{119}$$

where C_{D_o} and K are given by the Eqs. (117) and (118). Once the Mach number has been obtained the optimum altitude for a given weight is obtained from Eq. (79), or graphically from Fig. 7.

For the case of maximum endurance, as has been shown above,

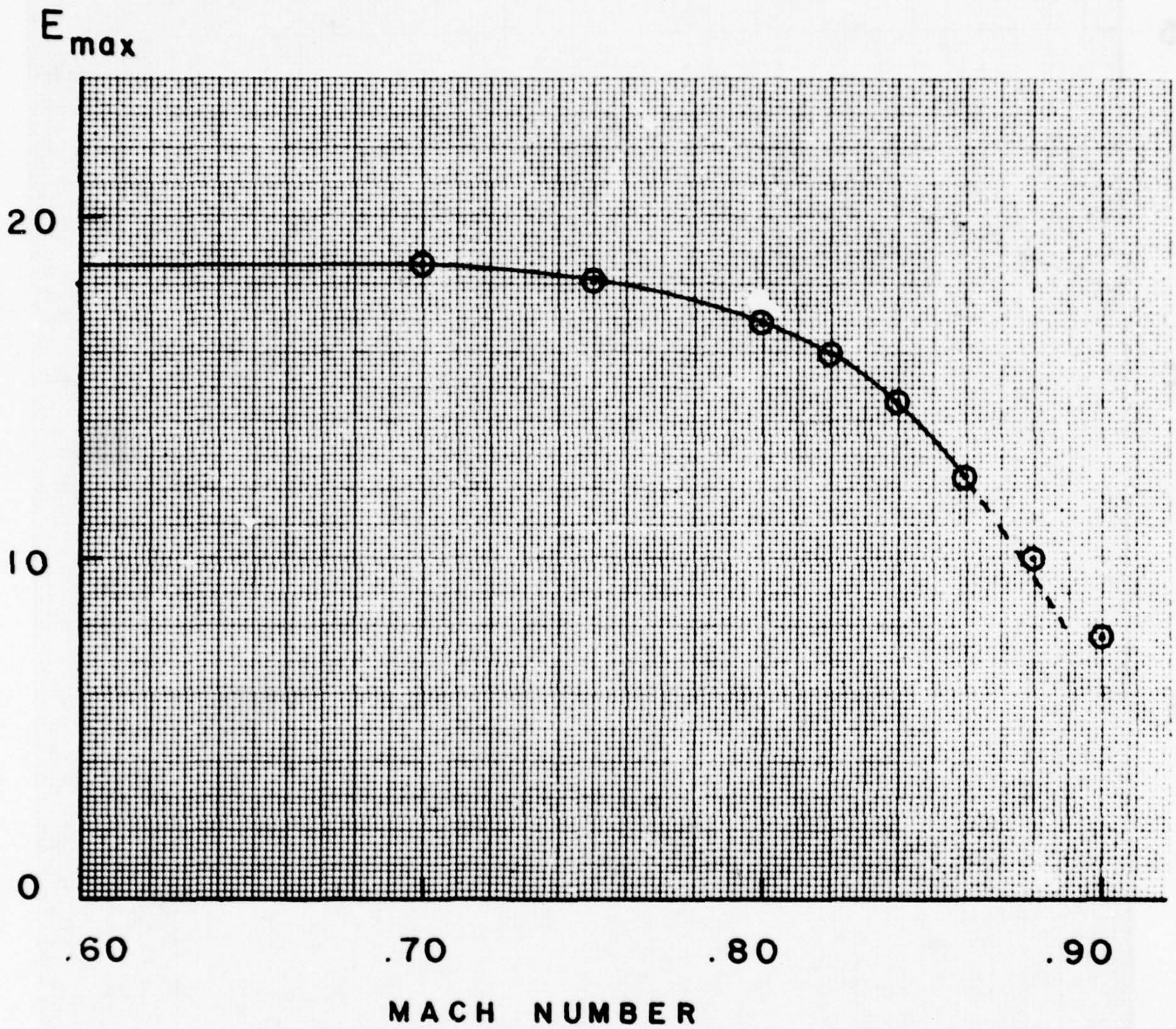

Fig. 10. Variation of E_{max} for the KC-135A.

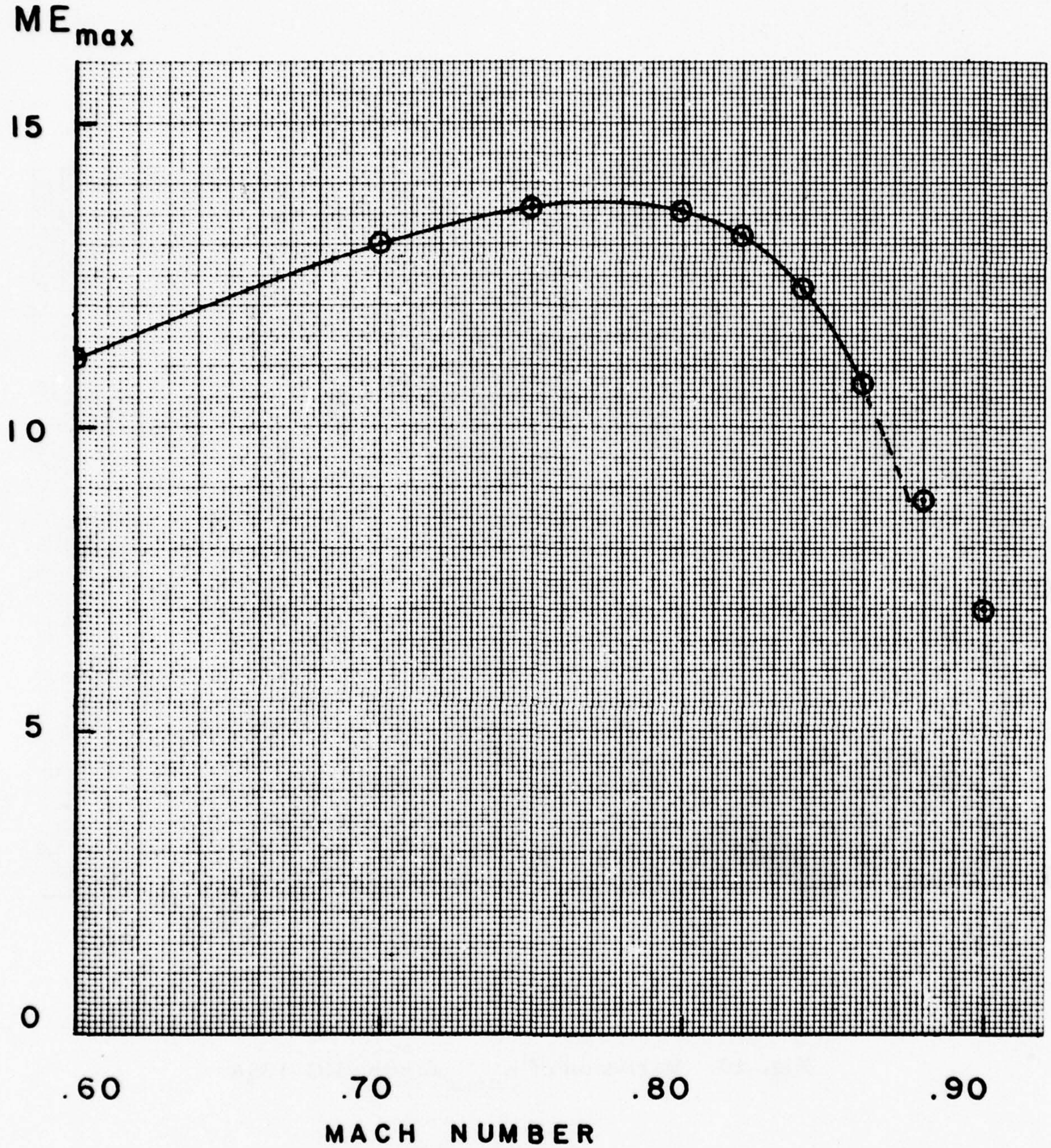

Fig. 11. Variation of ME_{max} for the KC-135A.

the optimum altitude is the highest one possible provided that $M \leq M_1$. Since the fuel consumption for climbing to the cruise altitude must be taken into consideration, and since the influence of the altitude on performance is small provided that the optimum Mach number is selected, the procedure to follow is to first select the altitude and then deduce the optimum Mach number from Eq. (96). This equation is plotted in Figure 12.

For the F-4C aircraft, compressibility effect is felt for $M > 0.6$ but stays small until $M = 0.7$. Hence, we shall take $M_1 = 0.70$. From flight test data, the following representations for C_{D_o} and K are accurate

$$C_{D_o} = 0.0154 + \frac{2.277 \times 10^{-3}(M-0.7)}{(0.9848-M)^{0.706112}} \tag{120}$$

$$K = 0.155203 + \frac{0.06573(M-0.7)}{(0.983-M)^{0.442726}} \tag{121}$$

The points selected for evaluating the formulas are the points M = 0.80, 0.90, 0.95.

For the range of Mach number of interest, between 0.7 and 0.95 the functions are in good agreement with flight test data as shown in Figures 13 and 14. Figure 15 plots E_{max} as function of the Mach number while Figure 16 presents the variation of $M\,E_{max}$. From the figure, it is seen that the optimum Mach number for maximum range is M = 0.85.

Numerical solution can be obtained by solving Eq. (119) with

C_{D_o} and K as given by Eqs. (120) and (121). Once the optimum Mach number has been obtained the optimum altitude for maximum range is given by Eq. (79) or graphically is obtained from Figure 7.

Finally, the function (96) for determining the optimum Mach number for maximum endurance is plotted in Figure 17 for the F-4C aircraft. For each selected altitude, and initial weight, we evaluate ω and then deduce the optimum Mach number from the graphs.

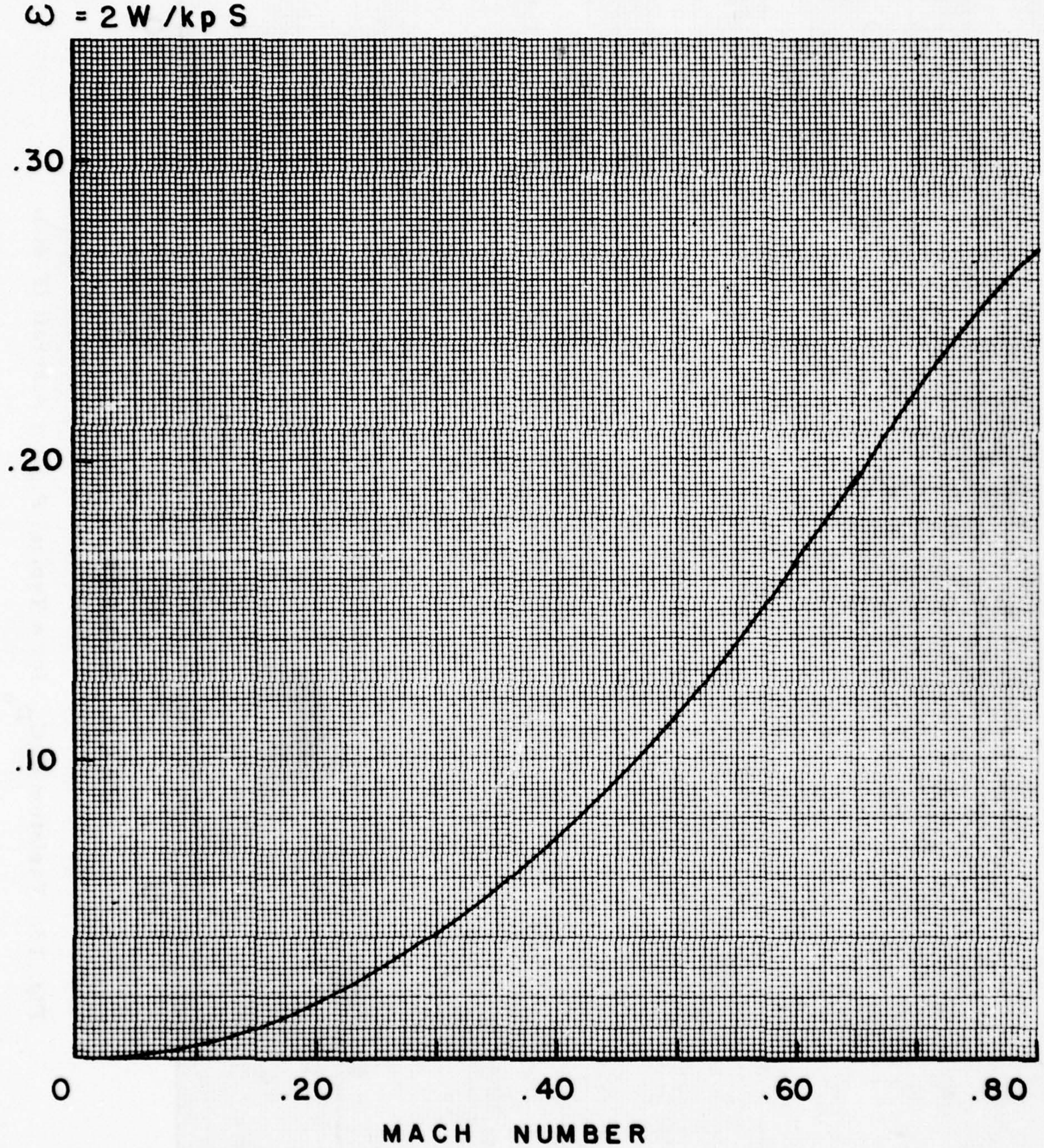

Fig. 12. Optimum Mach Number for Maximum Endurance as Function of the Altitude (KC-135A).

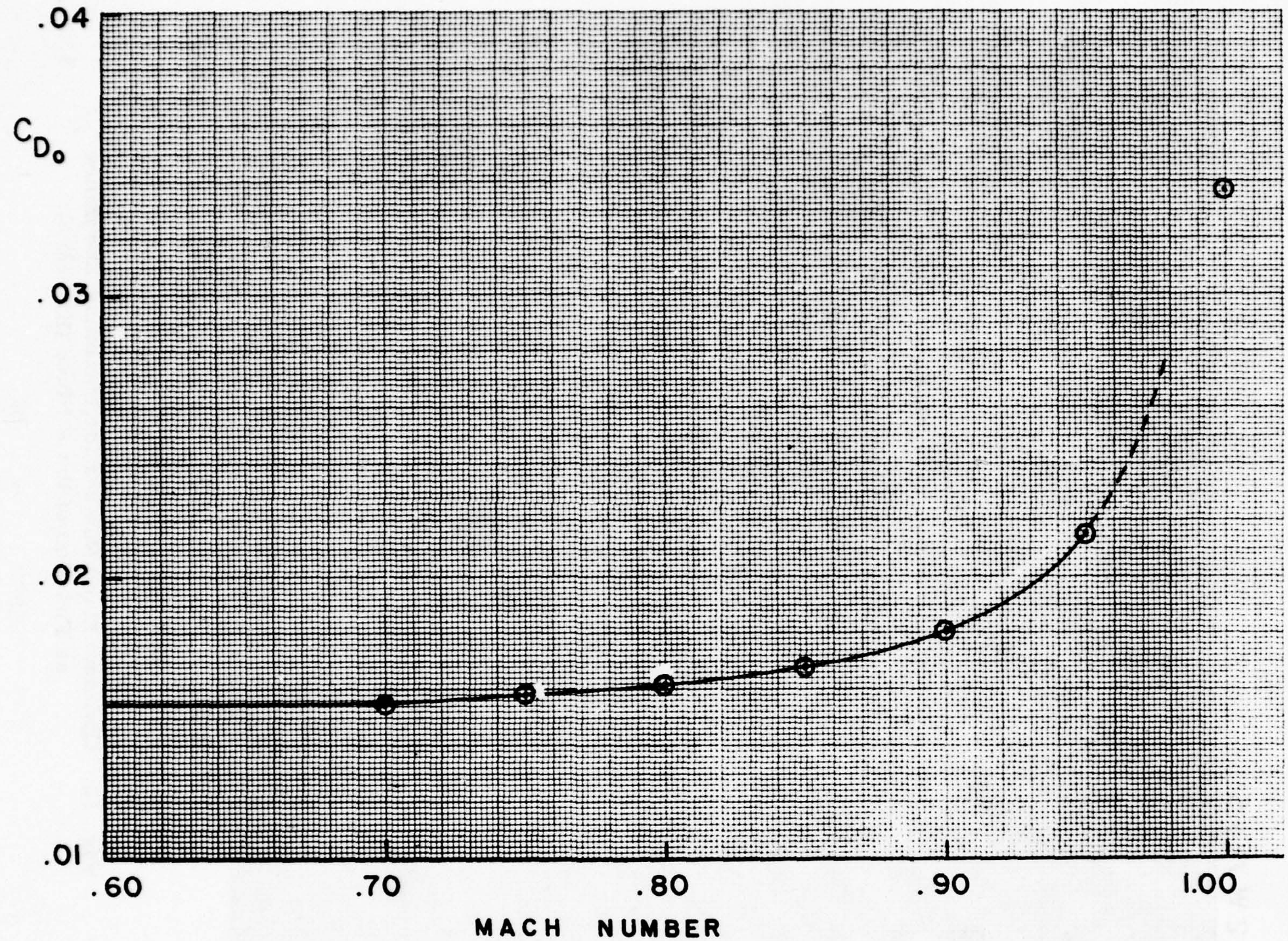

Fig. 13. Variation of C_{D_o} for a Typical Fighter Aircraft (F-4C).

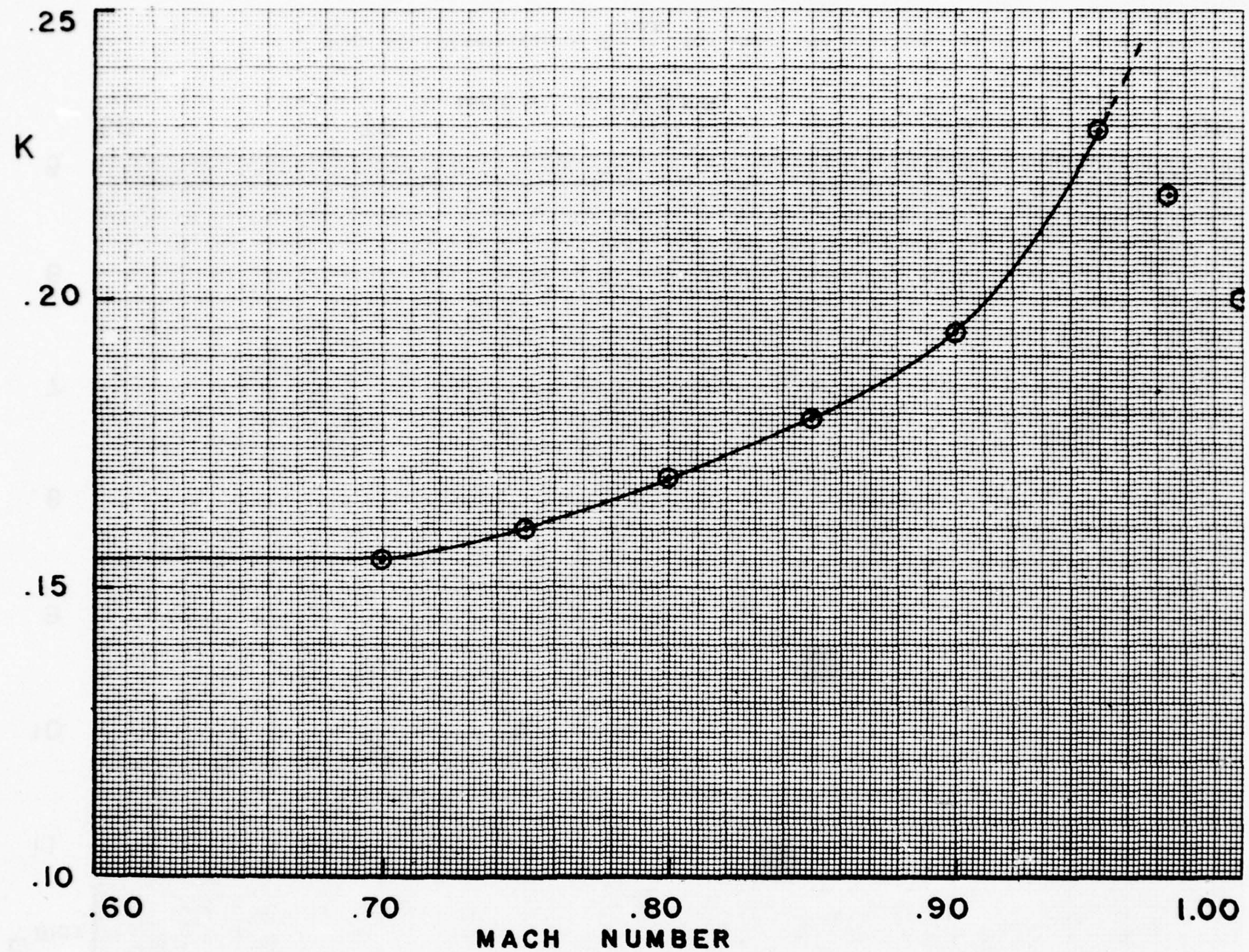

Fig. 14. Variation of K for a Typical Fighter Aircraft (F-4C).

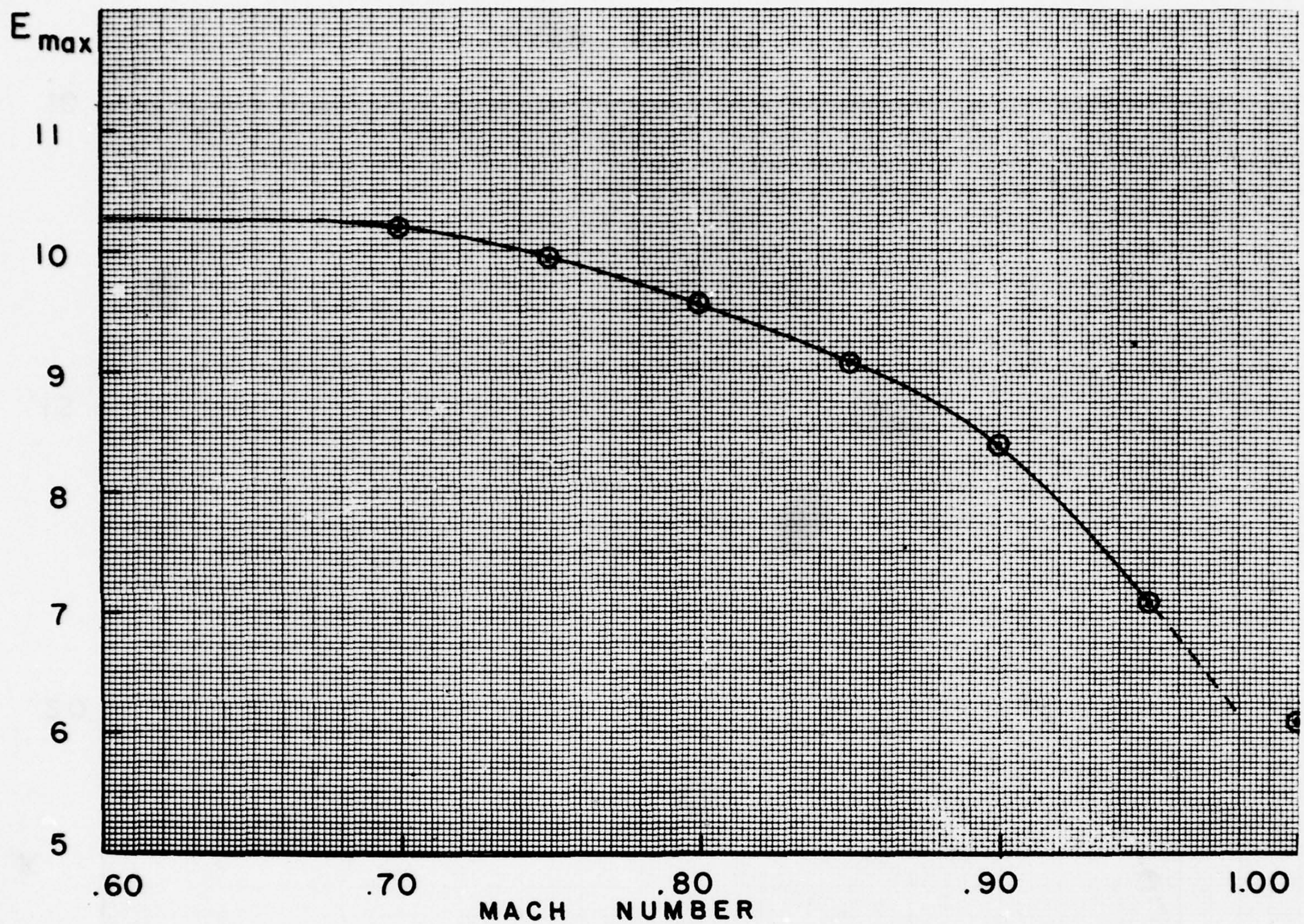

Fig. 15. Variation of E_{max} for the F-4C.

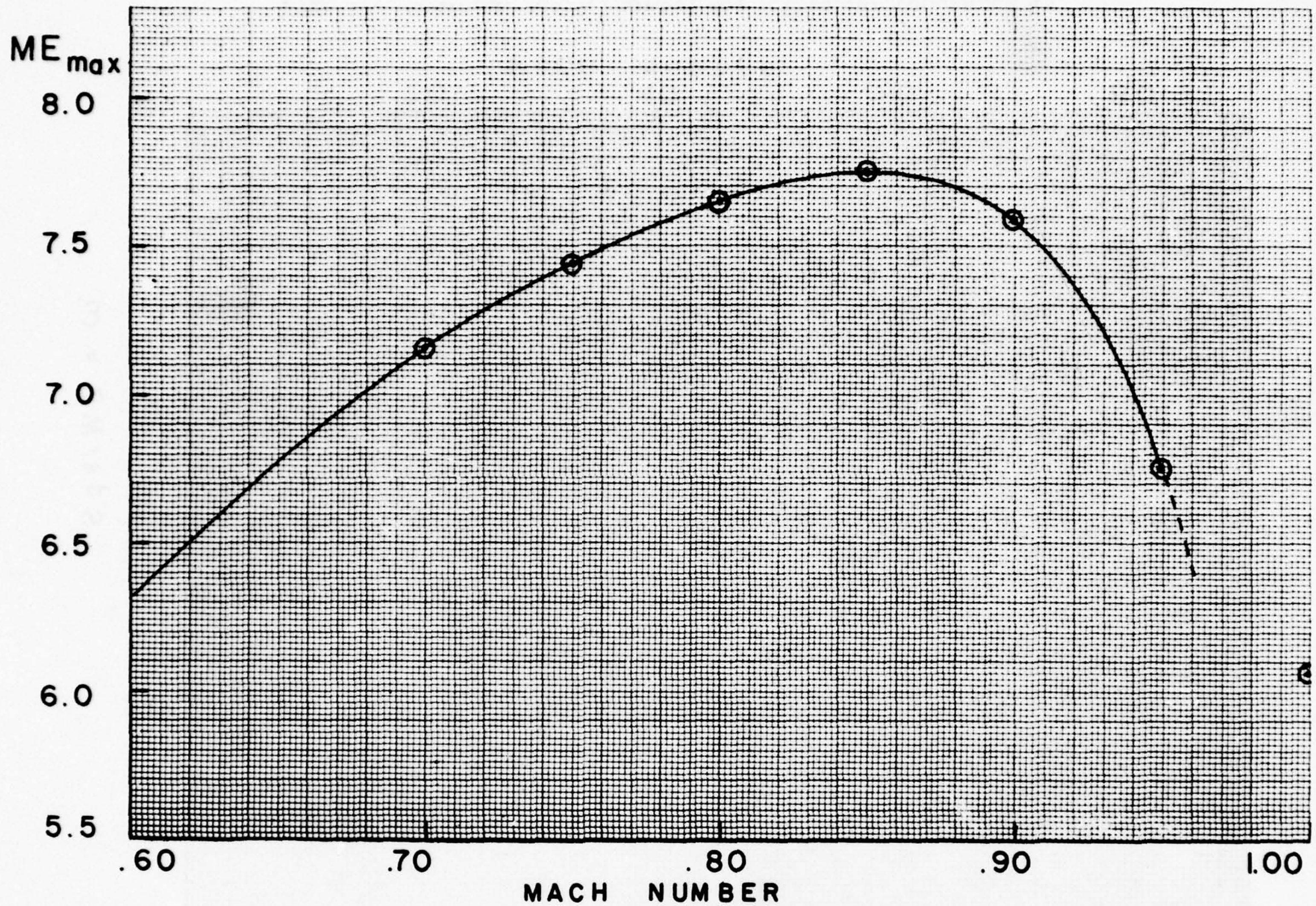

Fig. 16. Variation of ME_{max} for the F-4C.

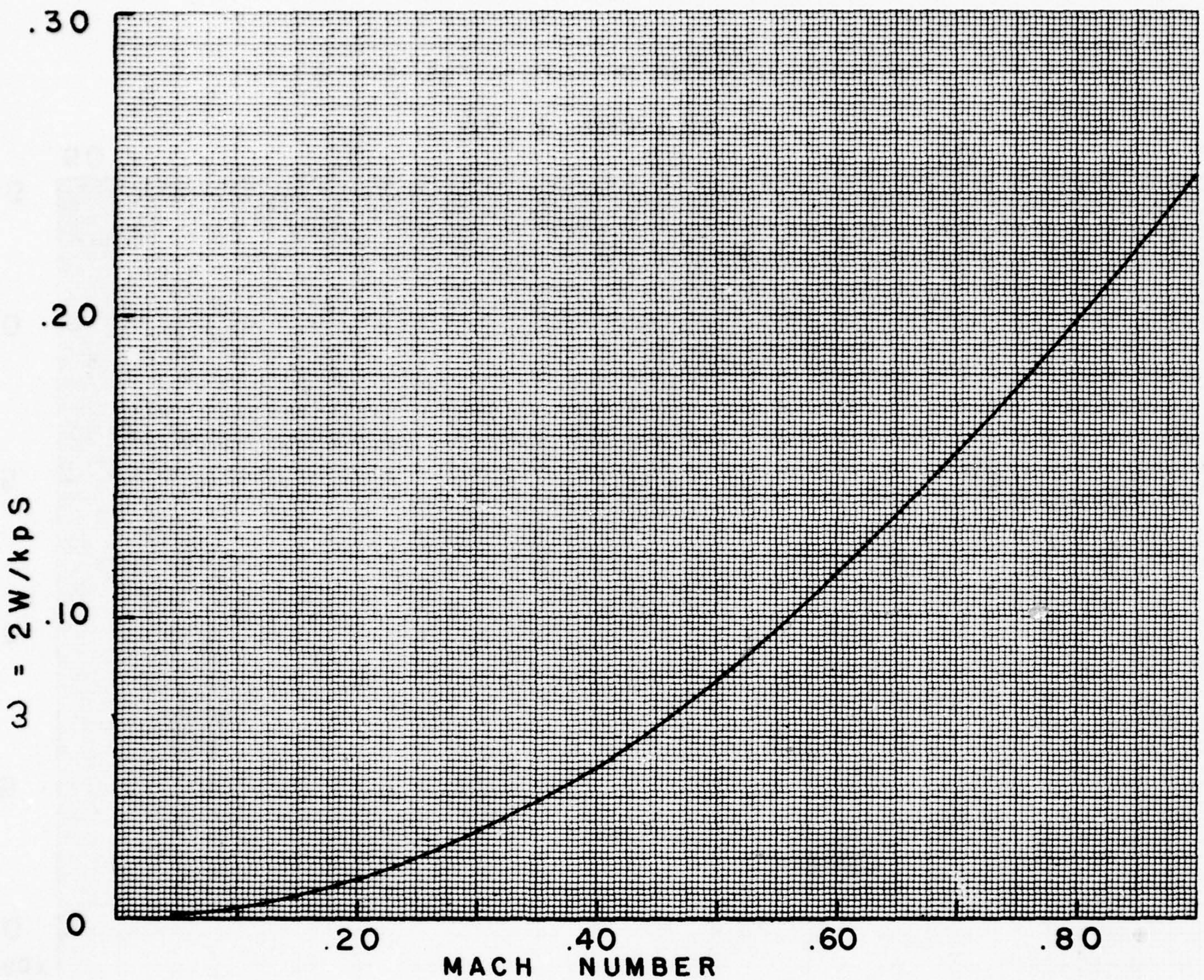

Fig. 17. Optimum Mach Number for Maximum Endurance as Function of the Altitude (F-4C).

SECTION VII

COMPLETE ANALYSIS OF THE CRUISE PERFORMANCE PROBLEM

In Section VI, we have used a rather engineering approach to derive the optimum lift and thrust controls for the cruise performance problem. The point which can raise objections is whether or not it is legitimate to use the altitude as a control. The arguments used are based on physical considerations rather than mathematical justifications. On the other hand, the advantage of the approach, if it is correct, is that we obtain explicit and simple law for the optimum lift control regardless of the end points while, as usually is the case with an optimal control problem, the control law can be obtained only after a two-point boundary value problem has been solved.

In this section, we will corroborate the results obtained by the simple analysis given in the previous section by considering the full set of equations.

Variational Formulation

The problem is formulated as an optimal control problem. The state variables are the dimensionless quantities x, η, M, γ and w. The initial values of the state variables are prescribed at the initial time $\theta = 0$. At the final time, $\theta = \theta_f$, the final values of the state variables are partially prescribed. It is proposed to find the optimal control law for the lift coefficient C_L and the dimensionless thrust τ to maximize x_f or θ_f.

First, we have the state equations

$$\frac{dx}{d\theta} = \eta^{(1-n)/2n} M$$

$$\frac{d\eta}{d\theta} = K_B \eta^{(n+1)/2n} M\gamma$$

$$\frac{dM}{d\theta} = \frac{\eta^{(n-1)/2n}}{w} [\tau - \eta^{-1} M^2 (C_{D_o} + KC_L^2)]$$

$$\frac{d\gamma}{d\theta} = \frac{\eta^{(n-1)/2n}}{w_M} (\eta^{-1} M^2 C_L - w) \tag{122}$$

$$\frac{dw}{d\theta} = - K_c \tau$$

$$\frac{d\theta}{d\theta} = 1$$

The Hamiltonian of the system is

$$H = p_\theta + p_x \eta^{(1-n)/2n} M + K_B p_\eta \eta^{(n+1)/2n} M\gamma - \frac{p_M M^2}{w} \eta^{-(n+1)/2n} (C_{D_o} + KC_L^2)$$
$$+ \frac{p_\gamma \eta^{(n-1)/2n}}{wM} (\eta^{-1} M^2 C_L - w) + \frac{\tau}{w} [\eta^{(n-1)/2n} p_M - wK_c p_w] \tag{123}$$

The control is the lift coefficient C_L and the dimensionless thrust τ subject to the constraints

$$0 \leq C_L \leq C_{L_{max}} (M) \tag{124}$$

and

$$0 \leq \tau \leq \tau_{max} (M) \tag{125}$$

The adjoint components are governed by the adjoint system

$$\frac{dp_\theta}{d\theta} = 0$$

$$\frac{dp_x}{d\theta} = 0$$

$$2n\eta \frac{dp_\eta}{d\theta} = (n-1)\eta^{(1-n)/2n} p_x M - (n+1)K_B p_\eta \eta^{(n+1)/2n} M\gamma -$$

$$(n+1)\eta^{-(n+1)/2n} \frac{M^2 p_M}{w}(C_{D_o} + KC_L^2) + \tag{126}$$

$$\frac{\eta^{-(n+1)/2n}}{w\,M} p_\gamma[(n+1)M^2 C_L + (n-1)\eta w] - (n-1)\eta^{(n-1)/2n} \frac{p_M}{w}\tau$$

$$M\frac{dp_M}{d\theta} = -p_x\eta^{(1-n)/2n} M - K_B p_\eta \eta^{(n+1)/2n} M\gamma - \frac{\eta^{(n-1)/2n}}{w} M p_\gamma(\eta^{-1}C_L + \frac{w}{M^2})$$

$$+ \frac{\eta^{-(n+1)/2n}}{w} M^2 p_M[C_{D_o}(2+C_{D_{o_M}}) + KC_L^2(2+K_M)] + K_c K_{c_M} p_w \tau$$

$$\gamma\frac{dp_\gamma}{d\theta} = -K_B p_\eta \eta^{(n+1)/2n} M\gamma$$

$$w\frac{dp_w}{d\theta} = -\frac{M^2 p_M}{w}\eta^{-(n+1)/2n}(C_{D_o} + KC_L^2) + \frac{p_\gamma \eta^{-(n+1)/2n}}{w} MC_L + \frac{\eta^{(n-1)/2n}}{w} p_M\tau$$

The problem is solved by integrating the state and adjoint equations, subject to the end conditions, while selecting at each instant θ the controls C_L and τ such that the Hamiltonian H is an absolute maximum.

With respect to C_L, H is a maximum either for $C_L = C_{L_{max}}$, or $C_L = 0$, or when $\partial H/\partial C_L = 0$. Explicitly, we have

$$2KC_L = \frac{p_\gamma}{Mp_M} \tag{127}$$

With respect to the linear thrust control τ, we define the switching function

$$\Theta = \eta^{(n-1)/2n} p_M - wK_c p_w \tag{128}$$

Then, when:

$\Theta > 0$, we choose $\tau = \tau_{max}$

$\Theta < 0$, we choose $\tau = 0$

$\Theta = 0$, for a finite time interval, we have τ = variable.

The variable thrust control is determined by singular control theory. We shall assume that the fuel weight $(W_i - W_f)$ is large enough so that we can use a suboptimum climb to cruise altitude and consider the flight as initiating with a sustaining arc along which τ is variable. Along this arc, we have constantly $\Theta = 0$, that is

$$\eta^{(n-1)/2n} p_M = wK_c p_w \tag{129}$$

Integrals of Motion

The problem has a number of integrals

$$p_\theta = C_1 \qquad p_2 = C_2 \tag{130}$$

The last equation in (122) is introduced so that we can write H = 0,

that is with the control law (127), and $\Theta = 0$

$$C_1 \eta^{(n+1)/2n} + C_2 \eta^{1/n} M + K_B p_\eta \eta^{(n+1)/n} M\gamma = \frac{M^2 p_M}{w}\left(C_{D_o} - KC_L^2 + \frac{2K\omega C_L}{M^2}\right) \tag{131}$$

This relation is valid along a coasting arc, $\tau = 0$, and along a variable thrust arc, $\Theta = 0$.

At this point, it is useful to summarize the problem. There are five state variables, χ, η, M, γ, w. At the initial time, we have

$$\theta = 0,\ x_i = 0,\ \eta_i,\ M_i,\ \gamma_i,\ w_i$$

At the final time we have

$$\theta_f,\ x_f,\ \eta_f,\ M_f,\ \gamma_f,\ w_f$$

For maximum range problem, we maximize x_f while allowing θ_f to be free. This leads to $C_1 = 0$ by the transversality condition. For maximum endurance problem, we maximize θ_f while allowing x_f to be free. This leads to $C_2 = 0$ by the transversality condition.

The system of state equations (122) is non-linear. Hence, optimum trajectory can only be obtained by numerical integration using some specified aerodynamic and engine data. The best we can hope for is to obtain, by some way, the optimum lift coefficient C_L and the optimum dimensionless thrust τ as function of the state variables so that upon substituting into the Eqs. (122), optimum trajectory

can be derived by numerical integration of the system.

In this respect, by Eq. (127), the lift coefficient can be expressed in terms of the state variables if p_γ and Mp_M can be obtained as functions of these variables. On the other hand, along the coasting arc and along the sustaining arc, by the Hamiltonian integral (131), C_L can be expressed in terms of the state variables if Mp_M and ηp_η can be expressed in terms of these variables.

Along the sustaining arc we have the additional relation (129) which, unfortunately does not contain either p_η or p_γ. Nevertheless, since the equation is valid over a finite time interval, which is the time interval for sustaining flight, we can take its derivative to have an additional relation. It is known that the linear control does not appear with this first derivative (Reference 4). By taking the derivative of Eq. (129) with respect to θ, and neglecting the effect of η, assumption which is rigorously exact for n=1, we have

$$C_2\eta^{1/n} M + K_B p_\eta \eta^{(n+1)/n} M\gamma = \frac{M^2 p_M}{w}\{C_{D_o}[2 + \eta^{(1-n)/2n} MK_c + (C_{D_o} K_c)_M] - KC_L^2[\frac{2\omega}{M^2 C_L} + \eta^{(1-n)/2n} MK_c - (KK_c)_M]\} \tag{132}$$

The three relations (129), (131) and (132) among wp_w, ηp_η and Mp_M allow the elimination of two of these variables. We need to find two more integrals, at least one of them involving γp_γ. In this respect, the non integrable equations of the system (126), with the use of the control law (127) and the singular relation (129), can be put into the form

$$2\eta\frac{d(\eta p_\eta)}{d\theta} = (n-1)\eta^{(1-n)/2n}C_2M + (n-1)K_Bp_\eta\eta^{(n+1)/2n}M\gamma -$$

$$(n+1)\eta^{-(n+1)/2n}\frac{M^2p_M}{w}[C_{D_o} - KC_L^2 - \frac{2(n-1)}{(n+1)}\frac{\omega KC_L}{M^2}] -$$

$$(n-1)\eta^{(n-1)/2n}\frac{p_M}{w}\tau$$

$$\frac{d(Mp_M)}{d\theta} = -C_2\eta^{(1-n)/2n}M - K_Bp_\eta\eta^{(n+1)/2n}M\gamma +$$

$$\frac{\eta^{-(n+1)/2n}}{w}M^2p_M[C_{D_o}(1+C_{D_{o_M}}) - KC_L^2(1-K_M) - \frac{2K\omega C_L}{M^2}] + \quad (133)$$

$$\frac{\eta^{(n-1)/2n}}{w}p_M(1+K_{C_M})\tau$$

$$\frac{d(\gamma p_\gamma)}{d\theta} = -K_Bp_\eta\eta^{(n+1)/2n}M\gamma + \frac{2\eta^{-(n+1)/2n}}{w}M^2p_M(KC_L^2 - \frac{\omega KC_L}{M^2})$$

$$\frac{d(wp_w)}{d\theta} = -\frac{\eta^{-(n+1)/2n}}{w}M^2p_M(C_{D_o} - KC_L^2)$$

These equations are valid for sustaining flight, and also for coasting flight with $\tau = 0$.

The only case where we can obtain an exact integral of this system is the case for flight in the stratosphere. With n=1, we have the equations in ηp_η and wp_w

$$\frac{d(\eta p_\eta)}{d\theta} = -\frac{M^2p_M}{w}(C_{D_o} - KC_L^2)$$

$$\frac{d(wp_w)}{d\theta} = -\frac{M^2p_M}{w}(C_{D_o} - KC_L^2) \quad (134)$$

Therefore

$$\frac{d(wp_w)}{d\theta} = \frac{d(\eta p_\eta)}{d\theta} \tag{135}$$

and upon integrating

$$wp_w = \eta p_\eta + C_3 \tag{136}$$

where C_3 is a new constant of integration

When $n \neq 1$, in order to obtain the required additional integrals some simplifying assumptions have to be made. If the optimum relation (132) for sustaining flight is used to eliminate the C_2 and K_B terms in the equation in Mp_M in system (133), we have

$$\frac{d(Mp_M)}{d\theta} = -\frac{M^3 K_c p_M}{\omega}(C_{D_o} - KC_L^2) + \frac{\eta^{-(n+1)/2n}}{w} p_M(1+K_{c_M})[\eta\tau - M^2(C_{D_o} + KC_L^2)] \tag{137}$$

Now if we assume that the Mach number stays nearly constant, that is at each instant of flight $dM/d\theta \simeq 0$, we have the assumption of quasi-unaccelerated flight

$$\eta\tau = M^2(C_{D_o} + KC_L^2) \tag{138}$$

Using this assumption, not as additional relation for providing the thrust law, but only as a simplifying device to obtain approximate integrals of the motion, the Eq. (137) is reduced to

$$\frac{d(Mp_M)}{d\theta} = -\frac{M^3 K_c p_M}{\omega}(C_{D_o} - KC_L^2) \tag{139}$$

Replacing K_c by its expression obtained from the singular relation (129)

$$\frac{d(Mp_M)}{d\theta} = -\frac{\eta^{(n-1)/2n}}{\omega}\frac{M^3 p_M{}^2}{wp_w}(C_{D_o} - KC_L^2) \tag{140}$$

Combining this equation with the equation for wp_w in system (133), we have

$$\frac{d(wp_w)}{wp_w} = \frac{d(Mp_M)}{Mp_M} \tag{141}$$

The integration yields

$$wp_w = Mp_M + C_4 \tag{142}$$

where C_4 is another constant of integration.

In system (133), by eliminating the K_B terms between the two equations for ηp_η and γp_γ we have

$$2n\frac{d(\eta p_\eta)}{d\theta} + (n-1)\frac{d(\gamma p_\gamma)}{d\theta} = (n-1)C_2\eta^{(1-n)/2n}M -$$

$$\frac{2n\eta^{-(n+1)/2n}}{w}M^2 p_M(C_{D_o} - KC_L^2) - \frac{(n-1)\eta^{-(n+1)/2n}}{w}$$

$$p_M[\eta\tau - M^2(C_{D_o} + KC_L^2)]$$

Combining this equation with the equation for wp_w, we have the exact differential relation

$$\frac{d(\eta p_\eta)}{d\theta} - \frac{d(wp_w)}{d\theta} + \frac{(n-1)}{2n}\frac{d(\gamma p_\gamma)}{d\theta} = \frac{(n-1)}{2n} C_2 \eta^{(1-n)/2n} M -$$

$$\frac{(n-1)\eta^{-(n+1)/2n}}{2nw} p_n[\eta\tau - M^2(C_{D_o} + KC_L^2)] \tag{146}$$

If n = 1, we can integrate the equation and have the exact integral (136). When $n \neq 1$, we can use the condition of unaccelerated flight (138) to reduce the equation to

$$\frac{d(\eta p_\eta)}{d\theta} - \frac{d(wp_w)}{d\theta} + \frac{(n-1)}{2n}\frac{d(\gamma p_\gamma)}{d\theta} = \frac{(n-1)}{2n} C_2 \eta^{(1-n)/2n} M \tag{144}$$

The equation can be integrated in the case of maximum endurance with free range, $C_2 = 0$. This gives

$$wp_w = \eta p_\eta + \frac{(n-1)}{2n}\gamma p_\gamma + C_3 \tag{145}$$

We need one more integral for the case of maximum range with free final time, $C_1 = 0$. This integral is not necessary to obtain the lift control law if we only consider the sustaining trajectory since when $C_1 = 0$ we can combine the two relations (131) and (132) to have

$$K[1 - \eta^{(1-n)/2n} MK_c + (KK_c)_M]C_L^2 - \frac{4K\omega}{M^2} C_L +$$

$$C_{D_o}[1 + \eta^{(1-n)/2n} MK_c + (C_{D_o}K_c)_M] = 0 \tag{146}$$

This quadratic equation can be solved for the optimum lift coefficient C_L in terms of the state variables η, M and w.

Applications of the Integrals

The constants of integration must be evaluated by using the boundary conditions. This requires a complete integration of the state and adjoint equations over the entire trajectory which may include a maximum thrust arc. An analytical integration of these equations is obviously not possible in the present case. Nevertheless, the integrals obtained are valuable because of the following reasons:

1. They reduce the order of the system. Each integral allows the elimination of one differential equation.

2. They may provide relations among the state variables or relations among the state and control variables. In the former case these relationships show explicityly that the optimum trajectory must lie on a certain surface. For example, in Section V, it is shown that optimum singular arc is the curve as given by Eq. (54). In the later case, the relations obtained give the optimum control as function of the state variables. This is illustrated by Eq. (146) above for the lift coefficient in the case of maximum range and later on in this section in Eq. (212) for the lift coefficient in the case of maximum endurance and in Eqs. (199)-(201) and (221)-(223) for the thrust control,respectively,in these cases.

3. When used with an independent numerical program, the exact integrals, if they exist, can be used to verify the accuracy of the numerical scheme.

4. Finally, in some favorable cases, the integrals obtained,

although not sufficient in number, may provide approximate optimum control law which can be used as a first estimate for an iteration procedure.

As illustration of this last statement we shall now use the integrals obtained to corroborate the results of the preceding section.

First, let us consider the case of flight in the stratosphere. Let

$$\begin{aligned} A &= M p_M \\ B &= w p_w \\ C &= \eta p_\eta \\ D &= \gamma p_\gamma \end{aligned} \tag{147}$$

where here A is different from the A defined above for A_R and A_E. For flight in the stratosphere, n=1, and we can write the Eqs. (129), (136), and (142) as

$$\begin{aligned} A &= MK_c\, B \\ B &= C + C_3 \\ B &= A + C_4 \end{aligned} \tag{148}$$

This linear system can be solved for A, B, C in terms of MK_c, that is, in terms of the Mach number, provided that $MK_c \neq 1$.

Now, in consistence with the assumptions used in Section VI we assume that the flight path angle γ not only is small but varies very slowly along the flight path. Hence, we have the equilibrium con-

dition

$$C_L = \frac{\omega}{M^2} \tag{149}$$

Using this condition, we rewrite the integrals (131) and (132) for n=1

$$\frac{C_1}{M} + C_2 + K_B C\gamma = \frac{A}{\omega}\left(C_{D_o} + \frac{K\omega^2}{M^4}\right) \tag{150}$$

and

$$C_2 + K_B C\gamma = \frac{A}{\omega}\{C_{D_o}[2 + MK_c + (C_{D_o}K_c)_M] - \frac{K\omega^2}{M^4}[2 + MK_c - (KK_c)_M]\} \tag{151}$$

The term in K_B is small because of the coefficient γ. Furthermore, it will be shown later that C is small, hence $K_B C\gamma$ can be neglected. We can also keep this term and assume that γ is constant, an assumption which is consistent with the equilibrium cruise condition. Then, since in the Eqs. (150) and (151), A and C are functions of the Mach number, these equations consist of a system of two equations for the variables ω and M. Hence, for flight in the stratosphere both the Mach number M and the wing loading ω are constant. These results agree with those in the previous section and are valid for both the maximum range and the maximum endurance problems. This should be noted here that they are only approximate. It will be shown later in a more rigorous analysis that the Mach number slowly decreases

along the flight path. From system (148), under the present assumptions, it appears that A, B and C are also constant. From any one of the equations for these variables, say Eqs. (134), we see that

$$C_L = \sqrt{\frac{C_{D_o}}{K}} = C_L^* \tag{152}$$

The flight for both the maximum range and the maximum endurance in the stratosphere is at maximum lift-to-drag ratio. From the equation for p_γ in system (126), we have, for n=1

$$\frac{dp_\gamma}{d\theta} = -K_B CM \tag{153}$$

On the other hand, from the optimum relation (127) for the lift coefficient

$$p_\gamma = 2KAC_L = \frac{A}{E_{max}} \tag{154}$$

The right hand side of this equation is function of the Mach number and therefore is approximately constant. Hence p_γ is near constant and from Eq. (153) it appears that the constant $C = \eta p_\eta$ is small and, as has been mentioned above, that $K_B C\gamma$ can be neglected.

For the case of maximum range with free final time, $C_1 = 0$, combining the two equations (150) and (151) we have

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}}\, A_R^{1/2} \tag{155}$$

For the case of maximum endurance with free range, $C_2=0$, using the Eq. (151) with the term $K_B C\gamma$ neglected, we have

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}}\, A_E^{1/2} \tag{156}$$

These results are in perfect agreement with those in Section VI.

Next, we consider the case of maximum endurance in the troposphere, $n \neq 1$. With A, B, C and D as defined in (147), we write the integrals (129), (142) and (145) which are valid in this case

$$A = \eta^{(1-n)/2n} MK_c B$$

$$B = A + C_4 \tag{157}$$

$$B = C + \frac{(n-1)}{2n} D + C_3$$

Also, we have the Eqs. (131) and (132), written with $C_2 = 0$ and using the equilibrium cruise condition (149)

$$\frac{C_1}{M} + K_B \eta^{(1-n)/2n} C\gamma = \frac{\eta^{(n-1)/2n}}{\omega} A(C_{D_o} + \frac{K\omega^2}{M^4}) \tag{158}$$

and

$$K_B \eta^{(1-n)/2n} C\gamma = \frac{\eta^{(n-1)/2n}}{\omega} A\{C_{D_o} [2 + \eta^{(1-n)/2n} MK_c + (C_{D_o} K_c)_M]$$
$$- \frac{K\omega^2}{M^4} [2 + \eta^{(1-n)/2n} MK_c - (KK_c)_M]\} \tag{159}$$

The optimum lift control law (127) is written in terms of A and D as

$$D = \frac{A\gamma}{E_{max}} \tag{160}$$

If γ is considered as constant, an assumption which is in line with the equilibrium cruise condition, then the Eqs. (157)-(160) constitute

a system of 6 equations for the unknowns A, B, C, D, M, ω, and $\eta^{(1-n)/2n}$. Hence, the first six variables are functions, of η, that is of the altitude. This influence of the altitude is small. For example, let us consider an altitude gain of about 2,000 ft, which is normally over a few hundred miles in range and in time interval long enough to insure the assumptions of equilibrium cruise and unaccelerated flight. Then, we have the ratio

$$\left(\frac{\eta_i}{\eta_f}\right)^{(1-n)/2n} = \left(\frac{p_i}{p_f}\right)^{(n-1)/2n} \tag{161}$$

With n = 1.235 and an altitude climb of 2,000 ft., this ratio varies from 1.008005 at 20,000 ft to 1.009154 at 36,000 ft. Hence, as a first order approximation we can neglect the effect of the altitude and deduce that the quantities A, B, C, and D are quasi-constant.

If A or B are considered as constant, then from their equations we deduce

$$C_L = \sqrt{\frac{C_{D_o}}{K}} = C_L^* \tag{162}$$

A result which is in agreement with the control law derived in Section VI. If D is considered as constant, from the equation in $\gamma p_\gamma = D$ in system (133) with the equilibrium cruise condition applied, we are led to neglecting the term $K_B \eta^{(1-n)/2n} CM\gamma$. Now, if $C = \eta p_\eta$ is considered as constant, then from the first of the Eqs. (133), using the condition for unaccelerated flight (138) and equilibruim cruise (149), we have

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$$(n-1)K_B\eta^{(1-n)/2n}C\gamma = \frac{2n\eta^{(n-1)/2n}}{\omega} A(C_{D_o} - KC_L^2) \tag{163}$$

From the fact that the left hand side is small, we have the control law (162).

Finally, we consider the case of maximum range with free final time, $C_1 = 0$, for flight in the troposphere, $n \neq 1$. As mentioned above there is one more integral to be found for this case. If the approximate condition (149) is used in the exact relation (146), we have

$$\frac{KC_L^2}{C_{D_o}} = \frac{1 + \eta^{(1-n)/2n}MK_c + (C_{D_o}K_c)_M}{3 + \eta^{(1-n)/2n}MK_c - (KK_c)_M} = A_R \tag{164}$$

Using the integral (131) with $C_1 = 0$ to eliminate the terms C_2 and K_B in the equation for ηp_η in system (133) and then applying the assumptions (138) and (149), we have

$$\frac{d(2n\eta p_\eta)}{d\theta} = -\frac{\eta^{-(n+1)/2n}}{w} M^2 p_M[(n+1)C_{D_o} - (3n-1)KC_L^2] \tag{165}$$

Combining this equation with the equation in wp_w in system (133)

$$\frac{d}{d\theta}[2n\eta p_\eta - (n+1)wp_w] = \frac{2(n-1)}{w}\eta^{-(n+1)/2n}M^2 p_M KC_L^2 \tag{166}$$

On the other hand, by subtracting the two equations in ηp_η and wp_w in system (133) using the same assumptions, we obtain

$$\frac{d}{d\theta}(\eta p_\eta - w p_w) = \frac{(n-1)}{2n}\eta^{(1-n)/2n} C_2 M + \frac{(n-1)}{2n} K_B p_\eta \eta^{(n+1)/2n} M\gamma \tag{167}$$

Again, if Eq. (131) with $C_1 = 0$ is used

$$\frac{d}{d\theta}[2n(\eta p_\eta - w p_w)] = (n-1)\frac{\eta^{-(n+1)/2n}}{w} M^2 p_M (C_{D_o} + KC_L^2) \tag{168}$$

Dividing the two equations (166) and (168) and noticing that

$$KC_L^2/C_{D_o} = A_R \tag{169}$$

we have

$$\frac{d[2n\eta p_\eta - (n+1)w p_w]}{d[2n(\eta p_\eta - w p_w)]} = \frac{2A_R}{1 + A_R} \tag{170}$$

As shown above, the effect of the altitude through the term $\eta^{(1-n)/2n} MK_c$ on the right hand side of this equation is negligible, not only that $\eta^{(1-n)/2n}$ is nearly constant but also because MK_c is negligibly small. Hence, the right hand side of the Eq. (170) is a function of the Mach number and if the condition of unaccelerated flight, M = constant, is applied solely for the purpose of performing the integration, we have

$$\frac{[(3n-1)A_R - (n+1)]}{(1 + A_R)} w p_w = \frac{2n(A_R - 1)}{(1 + A_R)} \eta p_\eta + C_5 \tag{171}$$

where C_5 is another constant of integration. This integral replaces the Eq. (145) for the case of maximum range in the troposphere.

We now can summarize the integrals of interest in this case, that is, the integrals (129), (142), and (171) as

$$A = \eta^{(1-n)/2n} MK_c B \qquad B = A + C_4 \tag{172}$$

$$\frac{[(3n-1)A_R - (n+1)]}{(1+A_R)} B = \frac{2n(A_R-1)}{(1+A_R)} C + C_5$$

where we recall that A_R is defined by Eq. (164). This relation results from combining the two integrals (131) and (132) so that we can write it with one of these integrals, say Eq. (131) with $C_1 = 0$ as

$$\frac{K\omega^2}{C_{D_o} M^4} = A_R \tag{173}$$

and

$$C_2 \eta^{(1-n)/2n} + K_B \eta^{(1-n)/2n} C\gamma = \frac{\eta^{(n-1)/2n}}{\omega} A \left(C_{D_o} + \frac{K\omega^2}{M^4}\right) \tag{174}$$

If the flight path angle γ is considered as constant, the equations (172), (173) and (174) constitute a system of 5 equations for the 6 variables A, B, C, ω, M, and $\eta^{(1-n)/2n}$. Hence, again we have the dependence of the first five variables on the altitude through $\eta^{(1-n)/2n}$. Since it has been shown that this term stays nearly constant, the variables A, B, and C are quasi-constant. If $C = \eta p_\eta$ is considered as constant, then from Eq. (165), we have the approximate optimum control law for the lift coefficient

$$C_L = \sqrt{\frac{n+1}{3n-1}} \sqrt{\frac{C_{D_o}}{K}} = \sqrt{\frac{n+1}{3n-1}}\, C_L^* \tag{175}$$

in perfect agreement with the law derived in Section VI for the case of maximum range in the troposphere. If we consider A or B as constant, then from their equations we have $C_L = C_L^*$. The effect of n

on the optimum lift coefficient as given by Eq. (175) is lost.

The rationale behind the selection of C as constant over A and B is as follows. From the first of the Eqs. (172), since $\eta^{(1-n)/2n} MK_c << 1$, we have $A << B$. By the same arguments as used above for flight in the stratosphere we can show that C is negligibly small. Then the term $K_B \eta^{(1-n)/2n} C\gamma$ in Eq. (174) is smaller and can be neglected. From this equation, since we maximize the range, we can normalize the adjoint variables by taking $C_2 = 1$. Then we can write the equation as

$$A = \frac{\omega \eta^{(1-n)/n}}{C_D} \qquad (176)$$

By this expression, A has a finite value not near zero. Therefore in decreasing order of magnitude, we have $B >> A >> C$ and it is most accurate to use C as a constant to obtain the optimum lift control law.

In conclusion, we notice that the maximum lift-to-drag ratio law and the constancy of the Mach number M and the wing loading ω are only approximate. The purpose of the analysis is simply to justify the engineering approach of Section VI. We shall next derive the main results of this report, namely obtaining the exact forms for the lift control and thrust magnitude for cruising flight to obtain either maximum range or maximum endurance.

<u>Singular Aerodynamics and Thrust Magnitude Controls for Maximum Range</u>

The thrust magnitude, as given by the condition (138) of unaccelerated flight, is only an approximation for the purpose of

simplifying the adjoint equations to make them integrable. To have the exact expression for the variable optimum thrust control, we start with the singular condition (129) rewritten here for convenience

$$\eta^{(n-1)/2n} p_M = K_c w p_w \tag{177}$$

This condition is identically satisfied along the main portion of the trajectory, namely the singular sustaining arc. By taking its derivative with respect to θ and neglecting the effect of η we have obtained the condition (132). If we retain the effect of η, then by taking the derivative of Eq. (177) using the state equations (122), the adjoint equations (126), the optimum lift condition (127) and the singular condition (177) itself, we have the following relation

$$C_2 + K_B \eta p_\eta \gamma + \frac{(1-n)}{2n} K_B M p_M \gamma = \frac{\eta^{-1/n}}{w} M p_M \{ C_{D_o} [2+\eta^{(1-n)/2n} M K_c + (C_{D_o} K_c)_M] - K C_L^2 [\frac{2\omega}{M^2 C_L^2} + \eta^{(1-n)/2n} M K_c - (K K_c)_M]\} \tag{178}$$

It is known that the linear thrust control does not appear in this derivative. Now, by replacing C_L by its optimum expression (127) in this equation and then by taking the derivative we will have a new relation in which the thrust magnitude τ will appear linearly. It should be noted that the optimum singular thrust control obtained this way is exact in the sense that the process does not involve any approximation except the approximation of small flight path angle

and results in the state equations (122), which we consider as exact equations forming the basis for the present analysis. It is only after the expression for the thrust has been obtained that we use the equilibrium cruise condition to simplify it for practical use. The process of taking the derivative of Eq. (178) is tedious but straightforward. Here we shall only consider the case of flight in the stratosphere. With n = 1, we have the simplified expression

$$C_2 + K_B \eta p_\eta \gamma = \frac{Mp_M}{\omega} \{C_{D_o} [2 + Mk_c + (C_{D_o} k_c)_M] - \frac{2k\omega}{M^2} C_L - KC_L^2 [Mk_c - (Kk_c)_M]\} \tag{179}$$

We notice that $K_c = k_c$ for n = 1.

On the other hand, we have the Hamiltonian integral (131) for n = 1

$$\frac{C_1}{M} + C_2 + K_B \eta p_\eta \gamma = \frac{Mp_M}{\omega} (C_{D_o} - KC_L^2 + \frac{2k\omega}{M^2} C_L) \tag{180}$$

For the case of maximum range with free final time, $C_1 = 0$, by combining the two equations (179) and (180) we have the quadratic equation for C_L

$$K[1 + (Kk_c)_M - Mk_c] C_L^2 - \frac{4K\omega}{M^2} C_L + C_{D_o} [1 + (C_{D_o} k_c)_M + Mk_c] = 0 \tag{181}$$

This equation, upon solving, gives the explicit expression for the optimum lift coefficient for maximum range in the stratosphere.

Let

$$K_1 = 1 + (Kk_c)_M - Mk_c$$
$$K_2 = 1 + (C_{D_o} k_c)_M + Mk_c \tag{182}$$

and write the equation for the optimum lift coefficient

$$KK_1 C_L^2 - \frac{4K\omega}{M^2} C_L + C_{D_o} K_2 = 0 \tag{183}$$

Solving for C_L

$$C_L = \frac{2\omega}{K_1 M^2} [1 \pm \sqrt{1 - \frac{K_1 K_2 C_{D_o}}{4K\omega^2} M^4}] \tag{184}$$

There is some inconvenience in using this formula. First, there is the ambiguity in the $(\pm)$ sign. If the singular arc starts at a point along a climb to the optimum cruise altitude, then the initial value for C_L is large and the (+) sign must be used. But each time the quantity under the square root passes through zero, we have to change the sign in front of the radical. Next, as can be seen in the optimum control law (127), the lift coefficient C_L is proportional to the adjoint component p_γ. Experience in numerical optimization has shown that this variable is nearly constant along the main portion of the flight trajectory, namely along the singular arc, where the thrust has a nearly constant intermediate level, but undergoes drastic variations at the two end-points. This can be interpreted as that the lift coefficient C_L varies nonlinearly at the two end-points in order to satisfy the initial and the final conditions but stay nearly constant along the main portion of the trajectory. For this type of behavior of the variable p_γ, the initial

value p_{γ_i}, that is the initial value C_{L_i}, is extremely sensitive to the subsequent trajectory generated. In other words, if the formula (184) is to be used, the exact initial point ω_i, M_i where the singular arc is initiated must be accurately evaluated in order to provide accurate initial value for C_L. Any small deviation from the exact initial lift coefficient C_{L_i} can result in steep climb or steep descent and hence not lead to the required final condition.

As will be seen later, the usefulness of the explicit equation (183) for the optimum lift coefficient C_L is that it is used to generate the optimum singular thrust control τ. As for the optimum lift coefficient itself, for all practical purposes, it is best to use the approximate control law

$$C_L = \sqrt{\frac{C_{D_o}}{K}} \qquad (185)$$

Using this control law, and the approximation of $d\gamma/d\theta \approx 0$, that is

$$\omega = C_L M^2 \qquad (186)$$

in the optimum relation (183), we obtain the approximate equation for evaluating the average optimum Mach number for cruising flight

$$K_1 + K_2 = 4 \qquad (187)$$

Explicitly, from Eqs. (182), we have

$$(KC_{D_o} k_c^2)_M = 2 \qquad (188)$$

For a given aircraft aerodynamic and engine characteristic, this

equation, upon solving, provides the average optimum Mach number in cruise. The optimum altitude for cruise is obtained as function of the optimum Mach number by combining the two equations (185) and (186)

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} \tag{189}$$

In summary, for maximum range, a first order optimum cruise solution can be obtained as follows. First, Eq. (188) is solved for the optimum cruise Mach number. Next the optimum cruise altitude is obtained from Eq. (189). The cruise trajectory at this altitude using constant optimum Mach number provides essentially the maximum range.

Constant Mach number cruise is only an approximation. Rigorously speaking, the optimum Mach number and hence, the optimum thrust coefficient, slowly decrease along the sustained flight path. To derive the optimum dimensionless thrust τ, we use the optimum relation (127) to rewrite the equation (183) as

$$MKK_1 p_\gamma^2 - 8K^2 \eta w p_\gamma p_M + 4K^2 K_2 C_{D_o} M^3 p_M{}^2 = 0 \tag{190}$$

Upon taking the derivative of this equation, the thrust magnitude will appear linearly. For convenience, we reproduce here the derivatives of the state and adjoint variables involved for the case n = 1

$$\frac{d\eta}{d\theta} = K_B \eta M \gamma$$

$$\frac{dM}{d\theta} = \frac{1}{\omega}[\eta\tau - M^2(C_{D_o} + KC_L^2)]$$

$$\frac{dw}{d\theta} = -k_c\tau \tag{191}$$

$$\frac{dP_\gamma}{d\theta} = -K_B \eta P_\eta M$$

$$\frac{dP_M}{d\theta} = -C_2 - K_B \eta P_\eta \gamma + \frac{MP_M}{\omega}\left[C_{D_o}(2 + C_{D_{o_M}}) + KK_M C_L^2 - \frac{2K\omega}{M^2} C_L\right] + \frac{k_{c_M} P_M}{\omega M}\eta\tau$$

Then, by taking the derivative of Eq.(190), using the Eqs. (191) and the Hamiltonian integral (180) with $C_1 = 0$, with the notation for logarithmic derivative, we have

$$\left[KK_1 P_\gamma^2 (1 + K_M + K_{1_M}) - \frac{16K^2 K_M \omega P_\gamma P_M}{M} + 4K^2 C_{D_o} K_2 M^2 p_M^2 (3 + 2K_M + C_{D_{o_M}} + K_{2_M})\right]$$

$$\times \frac{1}{\omega}[\eta\tau - M^2(C_{D_o} + KC_L^2)] - 8K^2(K_B \omega M\gamma - k_c \eta\tau) p_\gamma p_M - 2K_B KM(MK_1 p_\gamma - 4K\omega p_M)\eta p_\eta +$$

$$8K^2(C_{D_o} K_2 M^3 p_M - \omega p_\gamma) \times \left\{\frac{Mp_M}{\omega}\left[C_{D_o}(1 + C_{D_{o_M}}) + KC_L^2(1 + K_M) - \frac{4K\omega}{M^2} C_L\right] + \right.$$

$$\left.\frac{k_{c_M} p_M}{\omega M}\eta\tau\right\} = 0 \tag{192}$$

Since, by relation (127)

$$p_\gamma = 2KMp_M C_L \tag{193}$$

we can make the substitution, and upon simplifying by $4K^2 p_M^2$, we have

$$[KK_1M^2(1 + K_M + K_{1_M})C_L^2 - 8KK_M\omega C_L + K_2C_{D_o}M^2(3 + 2K_M + C_{D_{o_M}} + K_{2_M})]$$

$$x\frac{1}{\omega}[\eta\tau - M^2(C_{D_o} + KC_L^2)] - 4KMC_L(K_B\omega M\gamma - k_c\eta\tau) + K_BM^2(2\omega - K_1M^2C_L)\frac{\eta p_\eta}{Mp_M} +$$

$$\frac{2}{\omega}(C_{D_o}K_2M^2 - 2\omega KC_L)\ x[M^2C_{D_o}(1 + C_{D_{o_M}}) + KC_L^2M^2(1 + K_M) - 4K\omega C_L +$$

$$k_{c_M}\eta\tau] = 0 \qquad (194)$$

This expression is exact and can be solved for $\eta\tau$ which appears linearly in the equation.

For the ratio $\eta p_\eta/Mp_M$, we can use the exact singular relationship

$$Mp_M = Mk_c wp_w \qquad (195)$$

and the exact integral (136) for n=1

$$wp_w = \eta p_\eta + C_3 \qquad (196)$$

We also have the Hamiltonian integral (180) with $C_1 = 0$

$$C_2 + K_B\eta p_\eta\gamma = \frac{Mp_M}{\omega}(C_{D_o} - KC_L^2 + \frac{2K\omega}{M^2}C_L) \qquad (197)$$

Solving the system of Eqs. (195)-(197) for ηp_η and Mp_M and forming the ratio

$$\frac{\eta p_\eta}{Mp_M} = \frac{1}{\omega Mk_c(C_2/C_3 - K_B\gamma)}\ [\frac{C_2}{C_3}\omega - Mk_c(C_{D_o} - KC_L^2 + \frac{2K\omega}{M^2}C_L)] \qquad (198)$$

Upon substituting this expression into Eq. (194), and then replacing C_L by its expression as given in Eq. (184), we can solve for the optimum thrust control $\eta\tau$ in terms of the state variables M, ω, γ and one constant of integration C_2/C_3. In practice, it is accurate to neglect the term ηp_η, and also the term $K_B\gamma$ in Eq. (194). Then, the resulting expression for $\eta\tau$ depends solely on the two variables M and ω, as is the expression for the lift control C_L. Explicitly, we have

$$\eta\tau = \frac{P}{Q} \tag{199}$$

where

$$P = 4\omega M^2KC_L(2 - K_1 + K_{1_M})(C_{D_o} + KC_L^2) + K_2C_{D_o}M^4(K_{2_M} - K_{1_M})(C_{D_o} + KC_L^2)$$

$$+ K_2C_{D_o}M^4(K_1 + K_2)[(C_{D_o} + KC_L^2) - 2(Mk_c/K_2)KC_L^2] \tag{200}$$

and

$$Q = KK_1M^2C_L^2(2 - K_1 + K_{1_M}) + K_2C_{D_o}M^2(2 + K_2 + K_{2_M}) \tag{201}$$

In the expressions for P and Q, $KC_L{}^2$ can be expressed in terms of ωC_L by using Eq. (183), but it is more convenient to leave them in the present form. Then, since Mk_c/K_2 is small, it is seen that $\eta\tau$ is proportional to $C_D = C_{D_o} + KC_L^2$. If in P and Q, C_L is replaced by its approximate optimum expression (185) and the small term Mk_c/K_2 is neglected, we have the following approximate expression for the optimum singular thrust control

$$\eta\tau = \frac{8\omega\sqrt{KC_{D_o}}(2 - K_1 + K_{1_M}) + 2K_2C_{D_o}M^2(K_1 + K_2 + K_{2_M} - K_{1_M})}{K_1(2-K_1+K_{1_M}) + K_2(2 + K_2 + K_{2_M})} \tag{202}$$

Furthermore, if the equilibrium condition (186) is used as an approximation for ω we have the simplified expression

$$\eta\tau = \frac{4(2 - K_1 + K_{1_M}) + K_2(K_1 + K_2 + K_{2_M} - K_{1_M})}{K_1(2 - K_1 + K_{1_M}) + K_2(2 + K_2 + K_{2_M})}(2C_{D_o}M^2) \tag{203}$$

Finally, it has been said that the Mach number is not constant along the flight path. That is because the thrust magnitude $\eta\tau$ as given by the Eqs. (199)-(201) is slightly less than the drag. Using these equations, and the Eq. (183) for the lift coefficient in the equation for M, with $n = 1$, in system (122), we have after some manipulation

$$\frac{dM}{d\theta} = -\frac{2k_cC_{D_o}(K_1 + K_2)M^5KC_L^2}{\omega Q} \tag{204}$$

This equation is the exact equation for M. It is seen that the Mach number decreases at a slow rate because k_c is small. To have an idea about the reduction of the Mach number, we combine this equation with the equation for the weight to have

$$\frac{dw}{dM} = \frac{wP}{2C_{D_o}(K_1 + K_2)M^5KC_L^2} \tag{205}$$

If in the expression (200) for P we neglect the term Mk_c as compared to the drag coefficient and then use the approximate solutions

(185), (186) and (187), we have the approximate equation

$$\frac{dW}{W} = \frac{bdM}{M} \tag{206}$$

where

$$b = \frac{1}{4}[4(2 - K_1 + K_2) + K_1K_{1_M} + K_2K_{2_M}] \tag{207}$$

Holding b constant for the integration of Eq. (206) we have the approximate relationship

$$\frac{M_i}{M_f} = \left(\frac{W_i}{W_f}\right)^{1/b} \tag{208}$$

Figure 18 presents the plots of K_1, K_2 and $K_1 + K_2$ for the KC-135A using the functions for C_{D_o} and K as developed in Section VI. The point where $K_1 + K_2 = 4$ gives approximately the optimum Mach number for maximum range. At that point we have M = 0.774 and b = 8.938594. Likewise, Figure 19 presents the plots of the same functions for the F-4C using the functions for C_{D_o} and K as developed in Section VI. When $K_1 + K_2 = 4$, we deduce the approximate optimum Mach number M = 0.851 with the resulting value for b = 5.799053. Hence, in both cases, the reduction in the Mach number is relatively small as compared to the reduction in the weight.

Singular Aerodynamics and Thrust Magnitude Controls For Maximum Endurance

We now derive the complete solution for the maximum endurance problem. Again we restrict our consideration for flight in the stratosphere. For flight in the troposphere, the approach is the

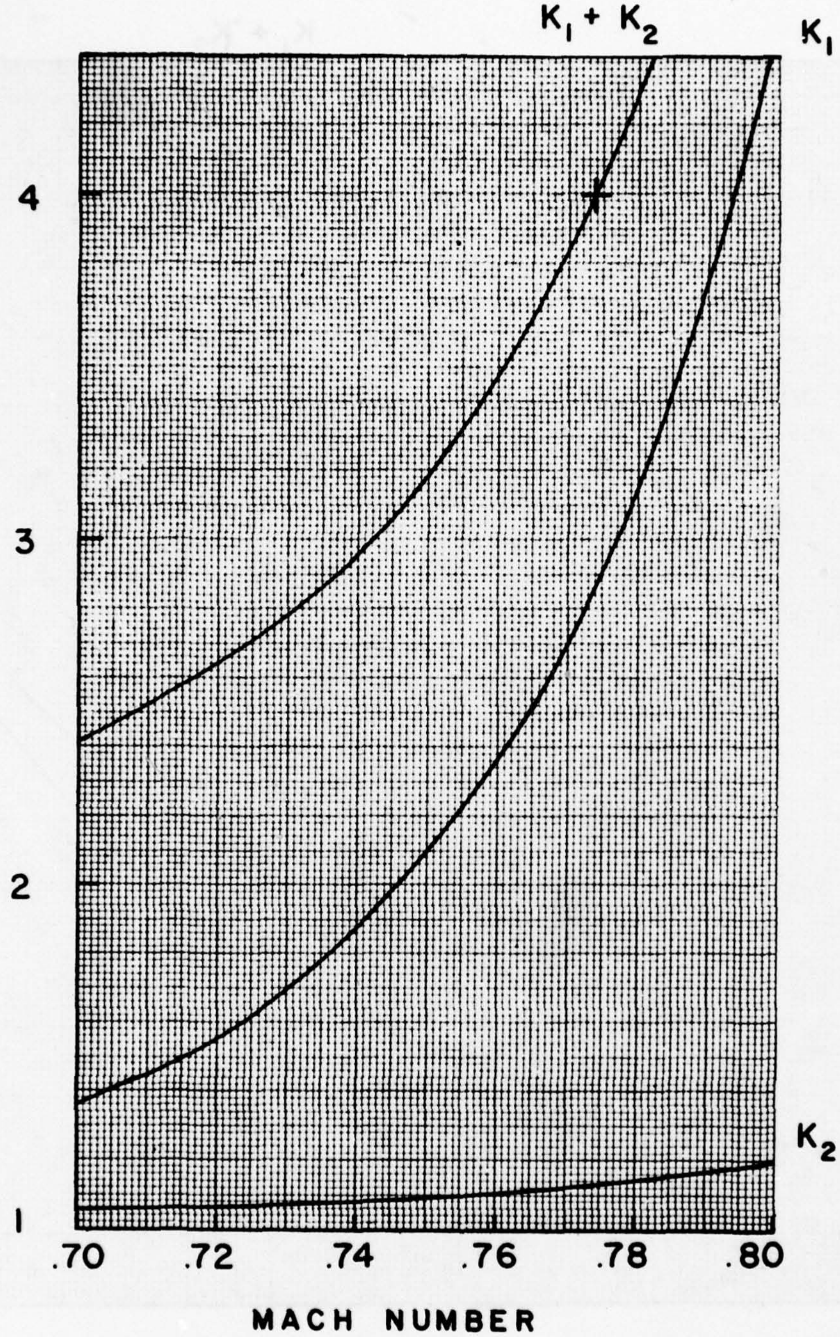

Fig. 18. Variations of the Functions K_1 and K_2 for the KC-135A.

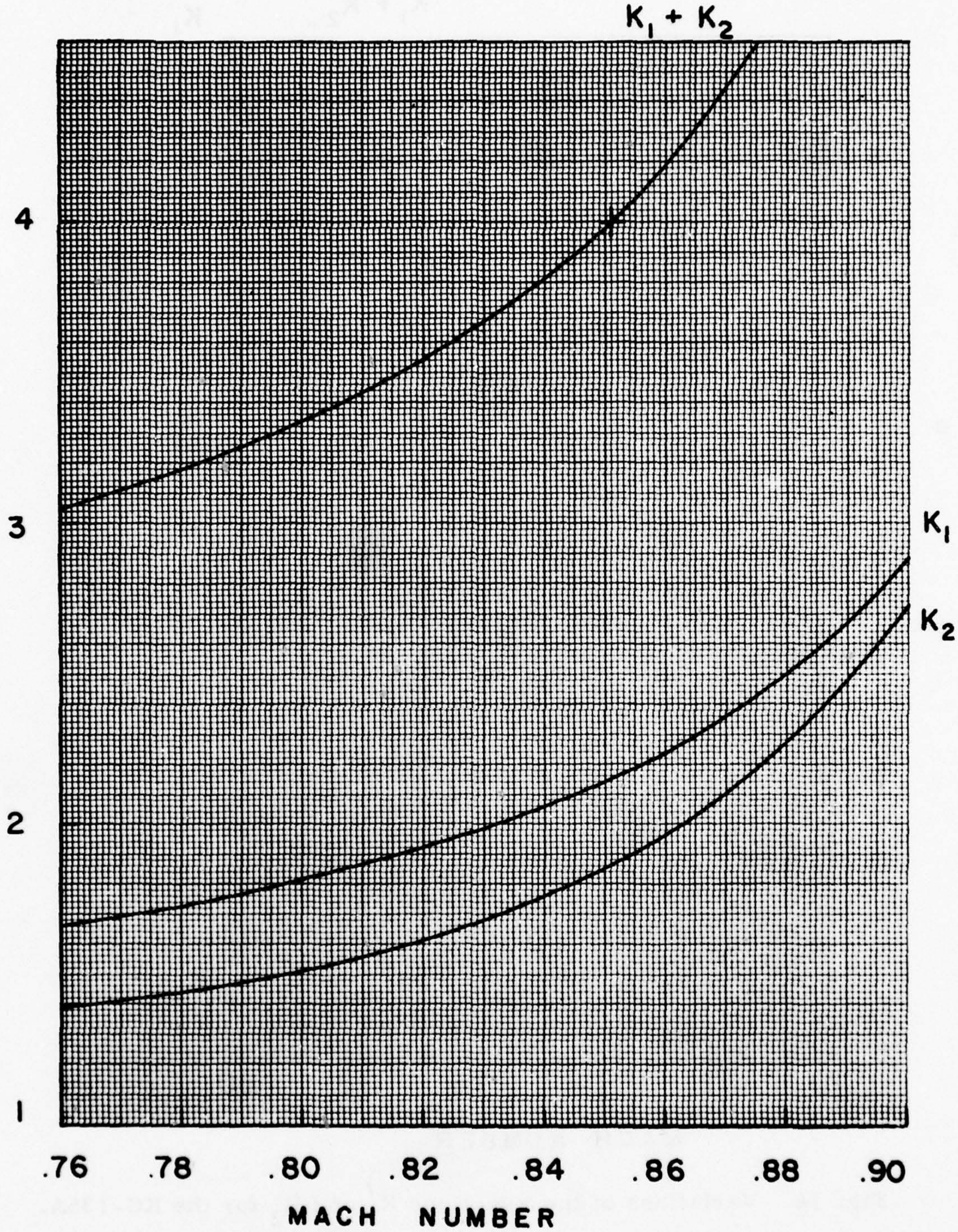

Fig. 19. Variations of the Functions K_1 and K_2 for the F-4C.

same but the derivation is more tedious and the resulting equations are more lengthy.

For the problem of maximum endurance, with free range, $C_2 = 0$, and the relations (179) and (180) become

$$K_B \eta p_\eta \gamma = \frac{Mp_M}{\omega} \{C_{D_o} [2 + Mk_c + (C_{D_o} k_c)_M] - \frac{2K\omega}{M^2} C_L \tag{209}$$

$$-KC_L^2 [Mk_c - (Kk_c)_M]\}$$

and

$$\frac{C_1}{M} + K_B \eta p_\eta \gamma = \frac{Mp_M}{\omega} (C_{D_o} - KC_L^2 + \frac{2K\omega}{M^2} C_L) \tag{210}$$

Rigorously speaking, by eliminating ηp_η, Mp_M among these equations and the combined relation from Eqs. (195) and (196)

$$Mp_M = Mk_c(\eta p_\eta + C_3) \tag{211}$$

we have a relation for evaluating the optimum lift coefficient C_L in terms of the variables M, ω, γ and the constant ratio C_1/C_3. Also by replacing C_L by $p_\gamma / 2KMp_M$ in Eq. (209) and then taking its derivative, we will have an equation in which the variable thrust magnitude $\eta\tau$ appears linearly, and thus can be solved in terms of the same variables and constant ratio. But in doing it this way, the expressions for the aerodynamics and thrust controls are unnecessarily long and the discussion of the way to evaluate the constant C_1/C_3 involved is not a simple matter. The simplest way, without compro-

mising the accuracy is to neglect the term $K_B \eta p_\eta \gamma$, which has been shown above to be very small, in Eq. (209). Then we immediately have the equation for the optimum lift control

$$KK_1 C_L^2 - \frac{2K\omega}{M^2} C_L + C_{D_o} K^2 = 0 \tag{212}$$

where this time, we define the function K_1 and K_2 as

$$\begin{aligned} K_1 &= (Kk_c)_M - Mk_c \\ K_2 &= 2 + (C_{D_o} k_c) + Mk_c \end{aligned} \tag{213}$$

Solving the quadratic equation (212) for C_L

$$C_L = \frac{\omega}{K_1 M^2} [1 \pm \sqrt{1 - \frac{C_{D_o} K_1 K_2}{K\omega^2} M^4}] \tag{214}$$

Maximum endurance is generally in the range of Mach number where compressibility effect is negligible. Hence while K_2 is always positive, K_1 is negative and small. The two roots of Eq. (214) are real. Regarding the sign ($\pm$) in the solution, it is seen that the (-) sign must be used. When K_1 is very small, we have

$$C_L = \frac{\omega}{K_1 M^2} [1 - (1 - \frac{C_{D_o} K_1 K_2}{2K\omega^2} M^4 + \ldots)]$$

or approximately

$$C_L = \frac{M^2 C_{D_o} K_2}{2\omega K} \tag{215}$$

Since by the equilibrium cruise condition $\omega/M^2 \approx C_L$ and at low Mach number $K_2 \approx 2$ we see that

$$C_L = \sqrt{\frac{C_{D_o}}{K}} = C_L^* \tag{216}$$

Hence the optimum flight for maximum endurance is conducted at near maximum lift-to-drag ratio. For better accuracy, the Eq. (214) with the (-) sign must be used.

To have the optimum singular thrust control, we use the optimum relation (127) to rewrite the equation (212) as

$$MKK_1 p_\gamma^{\,2} - 4K^2 \eta w p_\gamma p_M + 4K^2 K_2 C_{D_o} M^3 p_M^{\,2} = 0 \tag{217}$$

By taking the derivative of this equation, using the Eqs. (191) with $C_2 = 0$ we have the equation for the thrust magnitude control in sustaining flight.

As with the case of maximum range, we first have

$$\begin{aligned}
&[KK_1 p_\gamma^{\,2}(1 + K_M + K_{1_M}) - \frac{8K^2 K_M \omega p_\gamma p_M}{M} + 4K^2 K_2 C_{D_o} M^2 p_M^{\,2}(3 + 2K_M + \\
&C_{D_{o_M}} + K_{2_M})]\, x\frac{1}{\omega}[\eta\tau - M^2(C_{D_o} + KC_L^2)] - 4K^2(K_B \omega M\gamma - k_c \eta\tau) p_\gamma p_M \\
&-2K_B KM(MK_1 p_\gamma - 2K\omega p_M)\eta p_\eta + 4K^2(2K_2 C_{D_o} M^3 p_M - \omega p_\gamma) \\
&x\{\frac{Mp_M}{\omega}[C_{D_o}(2 + C_{D_{o_M}}) + K_M KC_L^2 - \frac{2K\omega}{M^2} C_L] + \frac{k_{c_M} p_M}{\omega M}\eta\tau\} = 0
\end{aligned} \tag{218}$$

By using relation (127) and simplifying by $4K^2 p_M^2$, we have

$$[KK_1M^2(1 + K_M + K_{1_M})C_L^2 - 4K_MK\omega C_L + K_2C_{D_o}M^2(3 + 2K_M + C_{D_{o_M}} + K_{2_M})]$$

$$\times\frac{1}{\omega}[\eta\tau - M^2(C_{D_o} + KC_L^2)] - 2KMC_L(K_B\omega M\gamma - k_c\eta\tau) + K_BM^2(\omega - K_1M^2C_L)\times$$

$$\frac{\eta p_\eta}{Mp_M} + \frac{2}{\omega}(K_2C_{D_o}M^2 - \omega KC_L)\ x[M^2C_{D_o}(2 + C_{D_{o_M}}) + K_MKM^2C_L - 2K\omega C_L +$$

$$k_{c_M}\eta\tau] = 0 \tag{219}$$

Concerning the ratio $\eta p_\eta/Mp_M$, it is obtained by solving the Eqs. (210) and (211). We have

$$\frac{\eta p_\eta}{Mp_M} = \frac{1}{\omega Mk_c(C_1/C_3M - K_B\gamma)}\left[\frac{C_1\omega}{C_3M} - Mk_c(C_{D_o} - KC_L^2 + \frac{2K\omega}{M^2}C_L)\right] \tag{220}$$

As is with the case of maximum range, we shall neglect this term and also the term $K_B\gamma$ in Eq. (219). Then, solving for $\eta\tau$ which appears linearly in the equation, we have

$$\eta\tau = \frac{P}{Q} \tag{221}$$

where this time, we have

$$P = 2\omega M^2KC_L(1 - K_1 + K_{1_M})(C_{D_o} + KC_L^2) + K_2C_{D_o}M^4(K_{2_M} - K_{1_M})(C_{D_o} + KC_L^2)$$

$$+ K_2C_{D_o}M^4(K_1 + K_2)[(C_{D_o} + KC_L^2) - 2(Mk_c/K_2)KC_L^2] \tag{222}$$

and

$$Q = KK_1M^2C_L^2(1 - K_1 + K_{1_M}) + K_2C_{D_o}M^2(1 + K_2 + K_{2_M}) \tag{223}$$

As with the case of maximum range, we use this optimum thrust law in the equation in M in system (122) with n=1 to have

$$\frac{dM}{d\theta} = -\frac{2k_cC_{D_o}(K_1 + K_2)M^5KC_L^2}{\omega Q} \tag{224}$$

This equation is the exact equation for M. Again it is seen that the Mach number decreases slowly. Combining this equation with the equation for w, we have

$$\frac{dw}{dM} = \frac{wP}{2C_{D_o}(K_1 + K_2)M^5KC_L^2} \tag{225}$$

If in the expression (222) for P, we neglect the small term Mk_c as compared to the drag coefficient, and then use the approximate solution (186), we have the approximate equation

$$\frac{dw}{w} = \frac{bdM}{M} \tag{226}$$

where now b is defined as

$$b = \frac{2(1 - K_1 + K_{1_M}) + K_2(K_{2_M} - K_{1_M}) + K_2(K_1 + K_2)}{(K_1 + K_2)} \tag{227}$$

Optimum Mach number for maximum endurance is low enough such that the effect of compressibility can be neglected. Hence we have approximately

$$b = 3 + 2Mk_c \tag{228}$$

and we write Eq. (226) as

$$\frac{dw}{w} = \frac{(3 + 2Mk_c)d(Mk_c)}{(Mk_c)} \tag{229}$$

Upon integrating from the initial point to the final point we have the approximate law for the reduction of the Mach number

$$\text{Log}\,\frac{W_i}{W_f} = 2k_c(M_i - M_f) + 3\,\text{Log}\,\frac{M_i}{M_f} \tag{230}$$

Finally, to show the effect of the altitude, we write the equation for the mass flow

$$\frac{dw}{d\theta} = -\frac{k_c}{\eta}\frac{P}{Q} \tag{231}$$

Now, if in the expressions (222) and (223) for P and Q, we retain the effect of Mk_c but using the condition of equilibrium cruise at maximum lift to drag ratio and neglecting the effect of the Mach number, we have after some manipulation

$$\frac{dw}{d\theta} = -\frac{wk_c}{E_{max}}\frac{(3 + Mk_c)}{(3 + 2Mk_c)} \tag{232}$$

Next, since $M = \eta^{1/2}w^{1/2}(K/C_{D_o})^{1/4}$, we have the equation

$$-\frac{dw}{d\theta} = \frac{wk_c[3 + k_c\eta^{1/2}w^{1/2}(K/C_{D_o})^{1/4}]}{E_{max}[3 + 2k_c\eta^{1/2}w^{1/2}(K/C_{D_o})^{1/4}]} \tag{233}$$

To minimize the mass flow we minimize the right hand side of this equation. At low subsonic speed where E_{max} is constant, this right hand side is a homographic equation in $\eta^{1/2}$ with negative derivative. Hence it decreases as $\eta^{1/2}$ increases. Since $\eta = 2 W_i/kpS$, we should use the minimum ambient pressure, hence conduct the flight at the highest altitude provided that $M \leq M_1$. Needless to say that Eq. (232) shows that the effect of the altitude is small and can only be detected by retaining the small term Mk_c, that is the second order term. Also the equation shows explicitly that high maximum *lift-to-drag ratio favors saving in fuel consumption.*

Finally, in Eq. (233), holding η constant and using constant Mach number, we can integrate the equation to have an approximate expression for the time of flight

$$\theta_f = \frac{2E_{max}}{k_c} \, Log \left\{ \frac{W_i[3 + k_c\eta^{1/2}(K/C_{D_o})^{1/4} w_i^{1/2}]}{W_f[3 + k_c\eta^{1/2}(K/C_{D_o})^{1/4} w_f^{1/2}]} \right\} \tag{234}$$

Again, in this explicit expression, it is seen that the right hand side increases with η. Hence, high altitude cruise favors endurance. But, if the cruise altitude is taken in the range where the optimum Mach number is greater than M_1 the effect of compressible fluid decreases E_{max} and the endurance decreases as the altitude increases.

A summary of the pertinent formulas derived in this section and the procedure for computing the optimum cruise trajectory are presented in Appendix B.

SECTION VIII

CONCLUSIONS

This report has presented the results of an extensive investigation of the cruise performance problem.

A typical mission profile consisted of a climb at high thrust setting to a certain optimum altitude, followed by a cruise at variable thrust until a final prescribed weight was nearly reached. The flight terminated by a descent at reduced thrust setting. For long range aircraft the main portion of the trajectory was the variable-thrust portion called the sustaining arc along which, using optimum control laws, significant savings in fuel consumption could be made.

It was proposed to find the variable thrust magnitude and the corresponding lift coefficient in sustaining flight such that for a prescribed final weight, either the range or the endurance was maximized.

The problems were first solved for the case of constant altitude flight for both the maximum range and the maximum endurance problems. It was shown that the influence of the Mach number on aerodynamics and engine characteristics was of prime consideration in evaluating the optimum Mach number.

Next, to improve the performance, the altitude was allowed to vary leading to the concept of cruise climb. Using the approximation of equilibrium cruise and quasi-unaccelerated flight, it was shown as a first order approximation that for both problems the op-

timum Mach number was nearly constant and the flight was at maximum lift-to-drag ratio. In the maximum range problem, there existed an optimum cruise altitude.

In the maximum endurance problem, the effect of the altitude on endurance of a high speed jet-propelled aircraft was small and could not be detected in the first order approximation. The problems were then solved using singular optimum control theory. For both the maximum range and the maximum endurance problems, the optimum lift coefficient C_L and the optimum variable thrust magnitude $\eta\tau$ were expressed explicitly in terms of the Mach number M and the dimensionless wing loading $\omega = 2W/kpS$. Hence, upon substitution into the equations of motion, optimum trajectory leading to maximum range or maximum endurance could be generated by numerical integration. It was shown that in both cases the optimum Mach number slowly decreases along the sustaining arc. Explicit formulas for evaluating the reduction in the Mach number were presented. In the case of maximum range this reduction depended upon the ratio of the final weight to the initial weight and the aerodynamic characteristics of the aircraft. In the case of maximum endurance the reduction in the Mach number depended upon the weight ratio only.

The influence of the altitude on the endurance was displayed explicitly by the rigourous analysis. It was shown that high maximum lift-to-drag ratio at low altitude cruise favored endurance.

A set of dimensionless variables were introduced to write the equations of motion in dimensionless form. Consequently, the re-

sults obtained were general and apply to any high speed, conventional, jet propelled aircraft.

In numerical applications, a modelling of the aerodynamics and engine characteristics was necessary. While in the applications, the specific fuel consumption coefficient was assumed independent of the Mach number, this effect was retained in the pertinent equations derived.

Two typical military aircraft were used in the applications, namely the transport-tanker KC-135A and the fighter F-4C. Functional modelling of the aerodynamic characteristics of these aircrafts was proposed and provides excellent agreement with flight test data in the range of Mach numbers of interest.

APPENDIX A

AERODYNAMIC CHARACTERISTICS

KC - 135A

The Boeing KC - 135A is a 4-engine jet-powered tanker-transport and is a military derivation of the Boeing prototype model 367-80.

The maximum lift coefficient $C_{L_{max}}$ and the buffeting lift coefficient C_{L_B} as function of the Mach number are given in Table 1.

Table 1

M	$C_{L_{max}}$	$C_{L_{max}} M^2$	C_{L_B}	$C_{L_B} M^2$
0.1	1.33	0.013	1.03	0.010
0.2	1.33	0.053	1.03	0.041
0.3	1.31	0.118	1.00	0.090
0.4	1.22	0.195	0.95	0.152
0.5	1.08	0.207	0.88	0.220
0.6	0.99	0.356	0.84	0.302
0.7	0.96	0.470	0.83	0.407
0.8	0.90	0.576	0.66	0.422
0.9	0.68	0.551	0.17	0.138

Drag polars for the KC-135A, as function of the Mach number, have been obtained from a correlation of wind tunnel tests and flight test results. From the drag polars, the values of C_{D_o} and K for $M \geq 0.7$ are deduced. The experimental data and the numerical values from mathematical modelling, Eqs. (A-1) and (A-2) are presented in Table 2.

Table 2

M	Experimental Data C_{D_o}	Experimental Data K	Numerical Modelling C_{D_o}	Numerical Modelling K
0.70	0.01230	0.05864	0.012300	0.05864
0.72			0.012317	0.05940
0.74			0.012338	0.06056
0.76			0.012365	0.06237
0.78	0.01240	0.06523	0.012400	0.06523
0.80	0.01245	0.06991	0.012450	0.06992
0.82	0.01254	0.07683	0.012530	0.07787
0.84	0.01290	0.09222	0.012641	0.09214
0.86	0.01350	0.11958	0.013500	0.11961

$$C_{D_o} = 0.0123 + \frac{2.29\text{x}10^{-4}(M - 0.7)}{(0.8665 - M)^{0.69339}} \tag{A-1}$$

$$K = 0.05864 + \frac{7.3\text{x}10^{-4}(M - 0.7)}{(0.9742 - M)^{2.884012}} \tag{A-2}$$

F - 4C

The maximum lift coefficient $C_{L_{max}}$ and the buffeting lift coefficient C_{L_B} as function of the Mach number for the fighter F-4C are given in Table 3

Table 3

M	$C_{L_{max}}$	$C_{L_{max}} M^2$	C_{L_B}	$C_{L_B} M^2$
0.70	0.95	0.4655	0.70	0.3430
0.80	0.95	0.6080	0.70	0.4480
0.90	0.94	0.7614	0.69	0.5589
1.00	0.88	0.8800	0.60	0.6000
1.10	0.80	0.9680	0.55	0.6655
1.20	0.78	1.1232	0.60	0.8640
1.30	0.66	1.1154	0.60	1.0140
1.40	0.55	1.0780	0.52	1.0192
1.50	0.44	0.9900	0.40	0.9000
1.60	0.32	0.8192	0.30	0.7680

$$C_{L_{max}} = 0.95$$

$$\text{for } M \leq 0.70$$

$$C_{L_B} = 0.70$$

For the F-4C, experimental values of C_{D_o} and K for $M \geq 0.7$ and their corresponding values from mathematical modelling, Eqs. (A-3) and (A-4) are presented in Table 4

Table 4

M	Experimental Data C_{D_o}	K	Numerical Modelling C_{D_o}	K
0.70	0.01540	0.155203	0.015400	0.155203
0.72			0.015516	0.157578
0.74			0.015646	0.160121
0.75	0.01572	0.161489	0.015717	0.161467
0.76			0.015792	0.162867
0.78			0.015958	0.165855
0.80	0.01615	0.169144	0.016150	0.169144
0.85	0.01675	0.178884	0.016806	0.179288
0.90	0.01800	0.194771	0.018000	0.194771
0.95	0.02150	0.229613	0.021490	0.229607

$$C_{D_o} = 0.0154 + \frac{2.277 \times 10^{-3}(M - 0.7)}{(0.9848 - M)^{0.706112}} \quad \text{(A-3)}$$

$$K = 0.155203 + \frac{6.573 \times 10^{-2}(M - 0.7)}{(0.983 - M)^{0.442726}} \quad \text{(A-4)}$$

APPENDIX B

COMPUTATION OF THE OPTIMUM TRAJECTORY IN SUSTAINED FLIGHT

Maximum Range

Equation for optimum Mach number

$$(KC_{D_o} k_c^2)_M = 2 \qquad \text{(B-1)}$$

Definition of logarithmic derivative

$$X_M = \frac{d(\text{Log } X)}{d(\text{Log } M)} = \frac{M}{X}\frac{dX}{dM} \qquad \text{(B-2)}$$

Equation for optimum cruise altitude

$$\omega = M^2 \sqrt{\frac{C_{D_o}}{K}} \qquad \text{(B-3)}$$

Optimum lift coefficient

$$C_L = \sqrt{\frac{C_{D_o}}{K}} \qquad \text{(B-4)}$$

First, compute the optimum Mach number from Eq. (B-1) and then the corresponding ω from Eq. (B-3). It is assumed that a suboptimum law for the climb portion of the trajectory has been used. Then the optimum cruise altitude is reached when the following condition is satisfied

$$\omega = \frac{2W}{kpS} \qquad \text{(B-5)}$$

The resulting weight $W = W_i$ is the initial weight for cruise and the resulting ambient pressure $p = p_i$ is the initial pressure, and hence the initial altitude for cruise.

The initial value of η is

$$\eta_i = \frac{2W_i}{kp_i S} \tag{B-6}$$

By integrating the equation of motion from $W = W_i$, $\gamma = \gamma_i = 0$, $\eta =$ η_i using constant optimum Mach number and the lift control law (B-4) until $W = W_f$ we obtain the maximum range in cruise.

Constant Mach number at maximum lift-to-drag ratio implies that the thrust law is

$$\eta\tau = 2C_{D_o} M^2 \tag{B-7}$$

To improve performance in the maximum range problem we shall use the following singular thrust control

$$\eta\tau = (2C_{D_o} M^2) \frac{4(2 - K_1 + K_{1_M}) + K_2(K_1 + K_2 + K_{2_M} - K_{1_M})}{K_1(2 - K_1 + K_{1_M}) + K_2(2 + K_2 + K_{2_M})} \tag{B-8}$$

where

$$\begin{aligned} K_1 &= 1 + (Kk_c)_M - Mk_c \\ K_2 &= 1 + (C_{D_o} k_c)_M + Mk_c \end{aligned} \tag{B-9}$$

If this law is used, the optimum Mach number slowly decreases along the sustained arc. In this case, the integration starts with an initial Mach number M_i slightly larger than the average optimum Mach

number computed from Eq. (B-1). This value M_i is a parameter to be optimized for best range.

Maximum Endurance

Optimum lift coefficient

$$C_L = \sqrt{\frac{C_{D_o}}{K}} \tag{B-10}$$

The altitude only has a secondary effect on endurance. It can be taken arbitrarily at low altitude where $M \leq M_1$ with M_1 being the Mach number beyond which the effect of compressible fluid being felt.

For the cruise arc, the integration starts with the initial condition

$$W = W_i, \ \eta = \eta_i = 2W_i/kp_iS, \ \gamma = \gamma_i = 0, \ M = M_i \tag{B-11}$$

The average value of the optimum Mach number at any given altitude is obtained from

$$\frac{2W_i}{kp_iS} = M^2 \sqrt{\frac{C_{D_o}}{K}} \tag{B-12}$$

The initial value M_i of the Mach number in Eq. (B-11) is selected slightly higher than the average value of M obtained from Eq. (B-12), and is a parameter to be optimized. It is to be selected to obtain the resulting best endurance.

The singular thrust control is given by Eqs. (221)-(223) which, upon simplification by using $C_L = \sqrt{C_{D_o}/K}$, $\omega = M^2 C_L$ and neglecting the small term Mk_c, become

$$\eta\tau = (2C_{D_o} M^2)\, \frac{2(1 - K_1 + K_{1_M}) + K_2(K_1 + K_2 + K_{2_M} - K_{1_M})}{K_1(1 - K_1 + K_{1_M}) + K_2(1 + K_2 + K_{2_M})} \tag{B-13}$$

where

$$K_1 = (Kk_c)_M - Mk_c$$
$$K_2 = 2 + (C_{D_o}k_c)_M + Mk_c \tag{B-14}$$

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